XX J-L. Lions Spanish French School
on Numerical Simulations in Physics & Engineering
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Data assimilation and integration into simulation models

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erc European Research Council
Established by the European Commission

IUF institut universitaire de France

école normale supérieure paris–saclay
UNIVERSITÉ PARIS-SACLAY

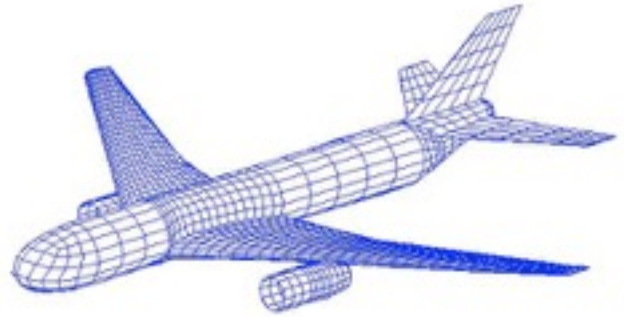
V&V Context

Physical model

- *in situ* data
- measurements

VALIDATION

« Do we solve the right equations? »



MASTERING VIRTUALIZATION

Numerical model

- FE discretization
- time discretization

IDENTIFICATION
OR UPDATING

VERIFICATION

« Do we solve the equations right? »

Mathematical model

- PDEs
- data: geometry/material/...

⬇

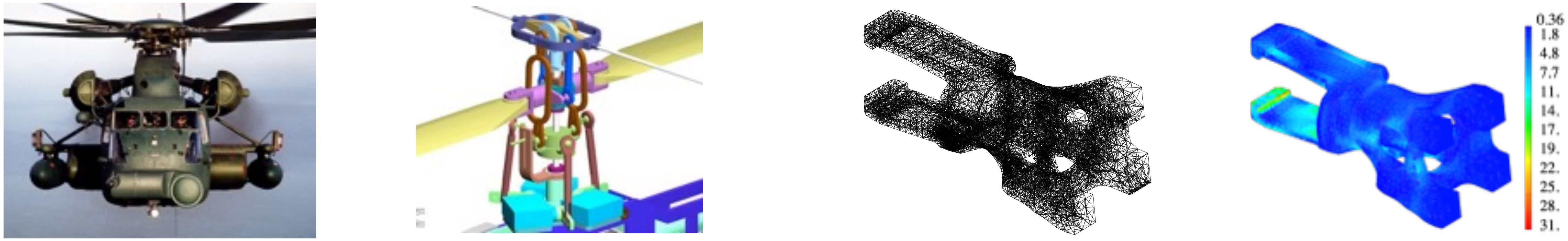
cascade of possible models

V&V: mandatory approach for reliable prediction

[Roarche 98, Oberkampf *et al* 03, Babuska & Oden 05]

Link between Data & Models

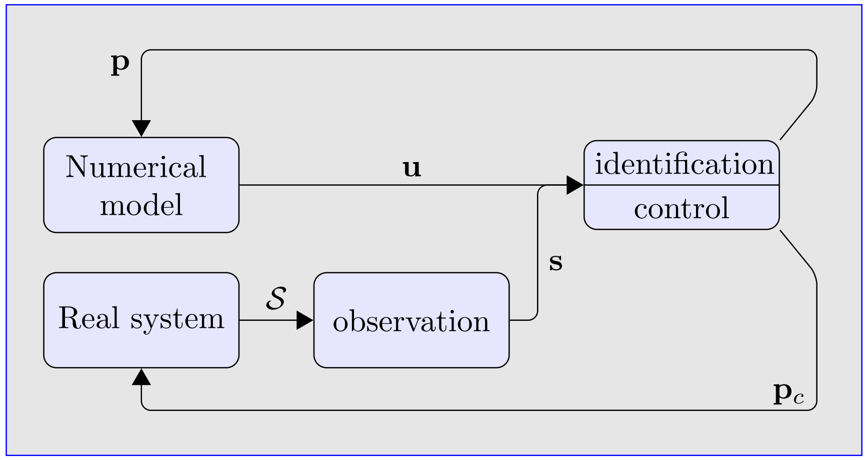
data → mathematical model → numerical model → *simulation outputs*



→ traditional vision of numerical simulation

→ difficult to take into account incomplete modeling (variabilities, uncertainties)

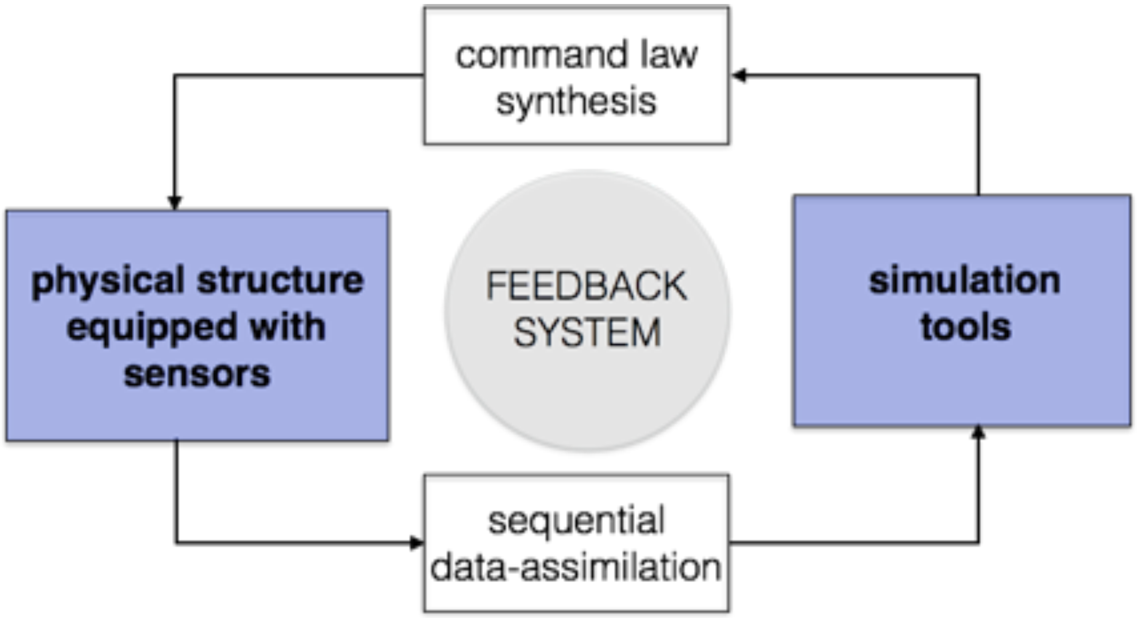
stronger interaction (synergy)
between data and models [Darema 04]



→ real-time feedback control between a system and its numerical model
connected systems, cyber-physics,...

Context

[Darema 04, 15]



command law synthesis

physical structure equipped with sensors

FEEDBACK SYSTEM

simulation tools

sequential data-assimilation

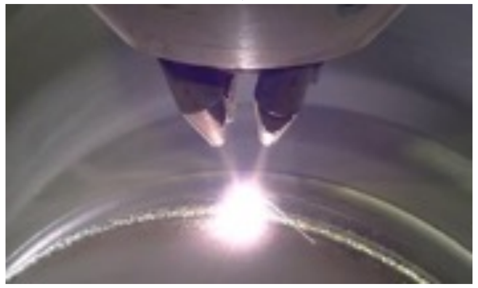
- customized digital twin to monitor evolving systems
- exploitation at best of models & data (knowledge)
- increased diagnosis & prognosis
  anticipation & interpretation
  decision-making
- economical & societal impact (performance, reliability)

Dynamic Data-Driven Application Systems (DDDAS)

assisted surgery

renewable energy

building performance



manufacturing

transportation

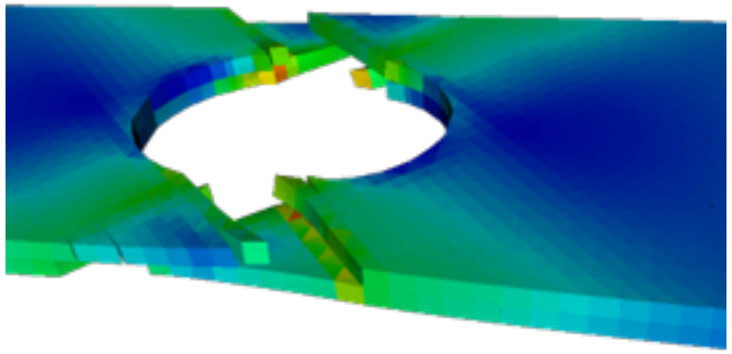
Context

- Application to **engineering structures** → **optimized maintenance**
  increased safety
  extended use (operation in degraded mode)

embedded strain sensors
(defects ∼ 1mm)

complex physics

↔



predictive simulations

→ *next-generation of integrated structural health monitoring (SHM)*
[Ding 07, Allaire 12, Prudencio *et al.* 15, Kapteyn *et al.* 20]

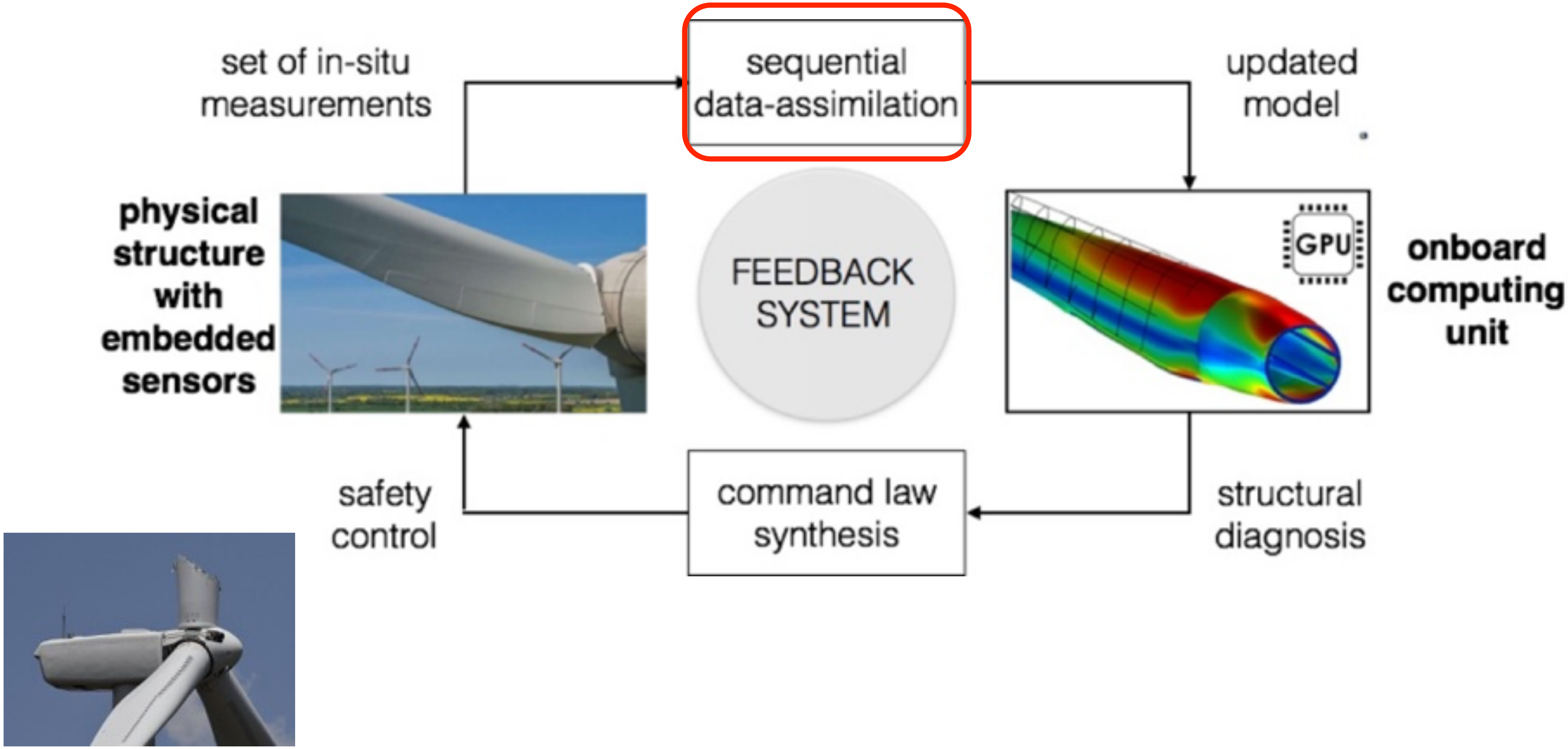
CHALLENGES

- **complex nonlinear** large systems with **uncertain environment**
- **real-time** requirements (early detection, reactivity)
- numerous, indirect and **noisy data**
- **biased models**

Real-time Data Assimilation

DDAS concept (numerical feedback loop) → requires optimal numerical procedures

- **Typical example: damage control**



Outline

1. Introduction to inverse problems
2. Deterministic inverse problems
3. Stochastic inverse problems
4. Sequential data assimilation
5. Some recent research applications

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1. Introduction to inverse problems
2. Deterministic inverse problems
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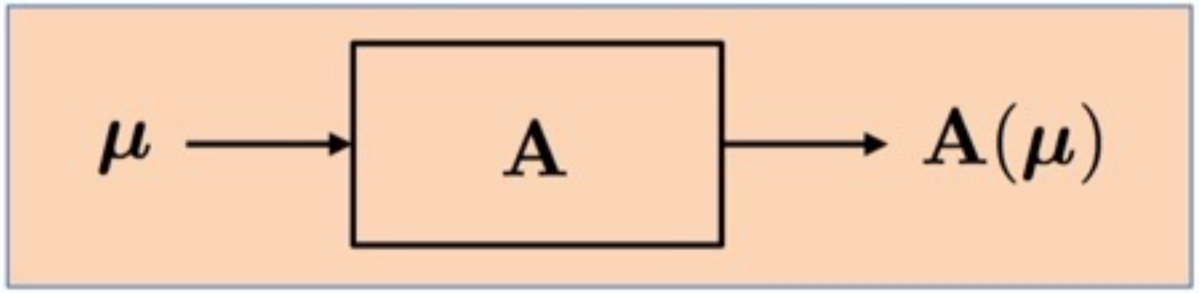
Mathematical Setting

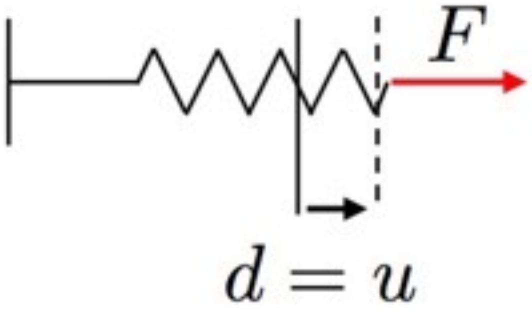
- Black-box model:

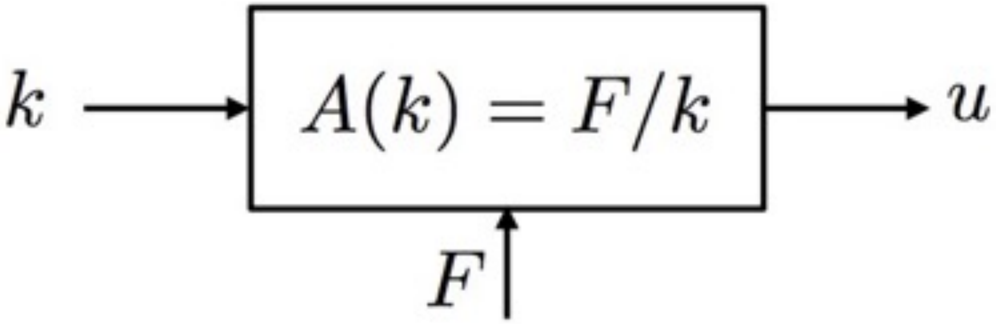
$$\begin{array}{rccc} \mathbf{A}: & \mathbb{R}^n & \mapsto & \mathbb{R}^m \\ & \boldsymbol{\mu} & \mapsto & \mathbf{A}(\boldsymbol{\mu}) \end{array}$$



- Inverse problem: Given $\mathbf{d} \in \mathbb{R}^m$, find $\boldsymbol{\mu}^\star \in \mathbb{R}^n$ such that $\mathbf{A}(\boldsymbol{\mu}) = \mathbf{d}$

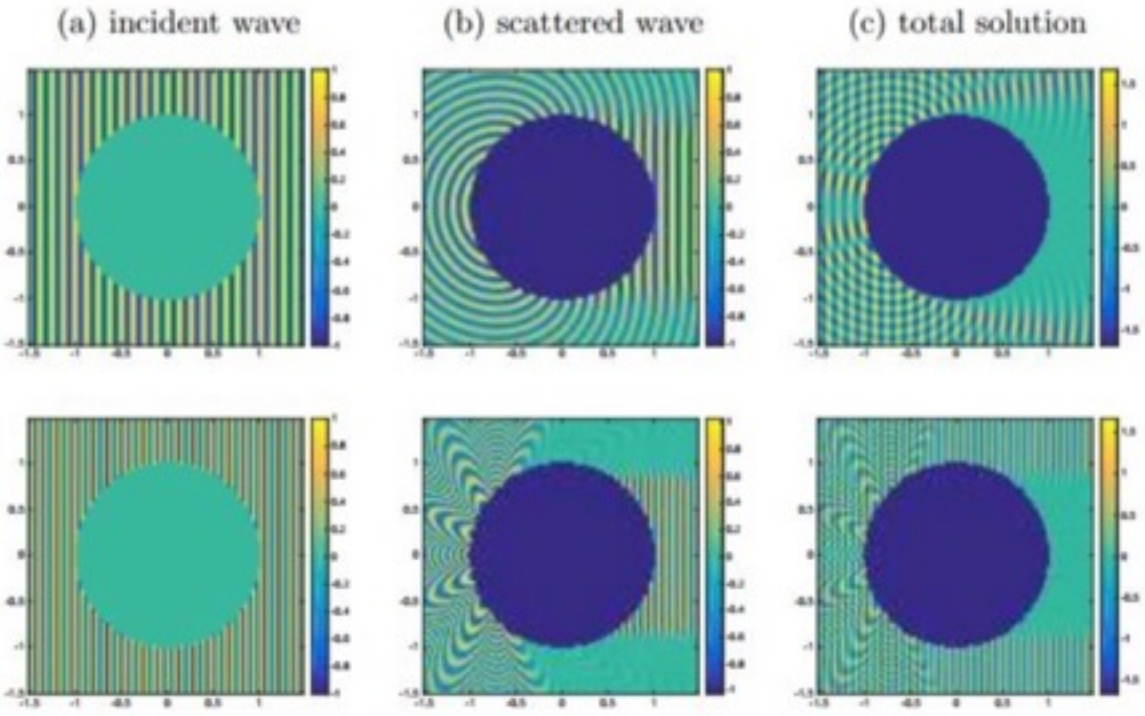
- Example:





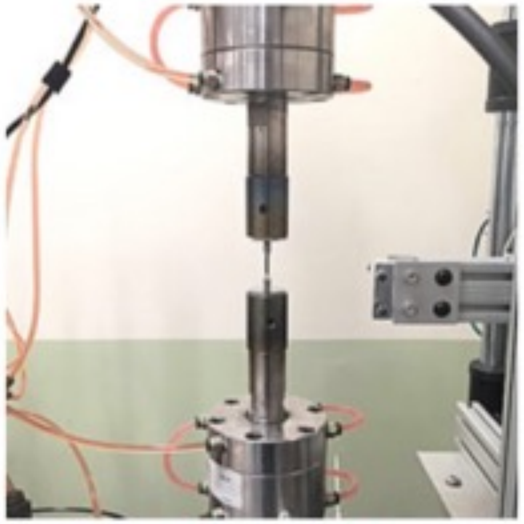
Inverse Scattering (radar, MRI)

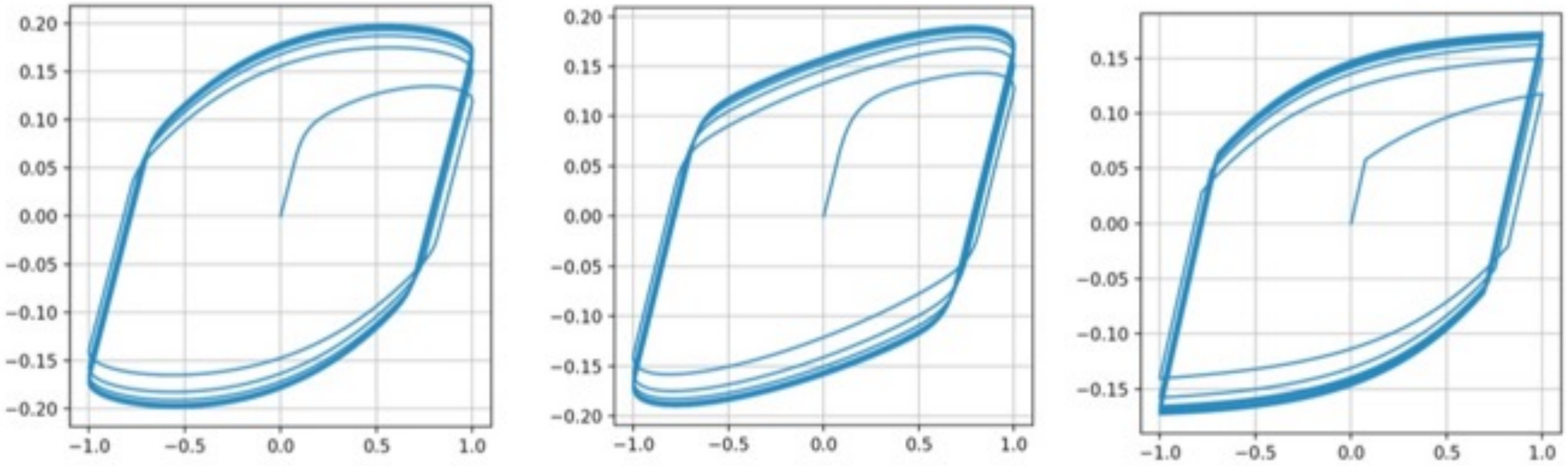
- Forward model: incident rate (and other boundary conditions) + wave propagation laws
- Measurement: scattered wave
- Find geometry and position of object



- (You can now make predictions (likely trajectories, …) using the results of the inverse problem)

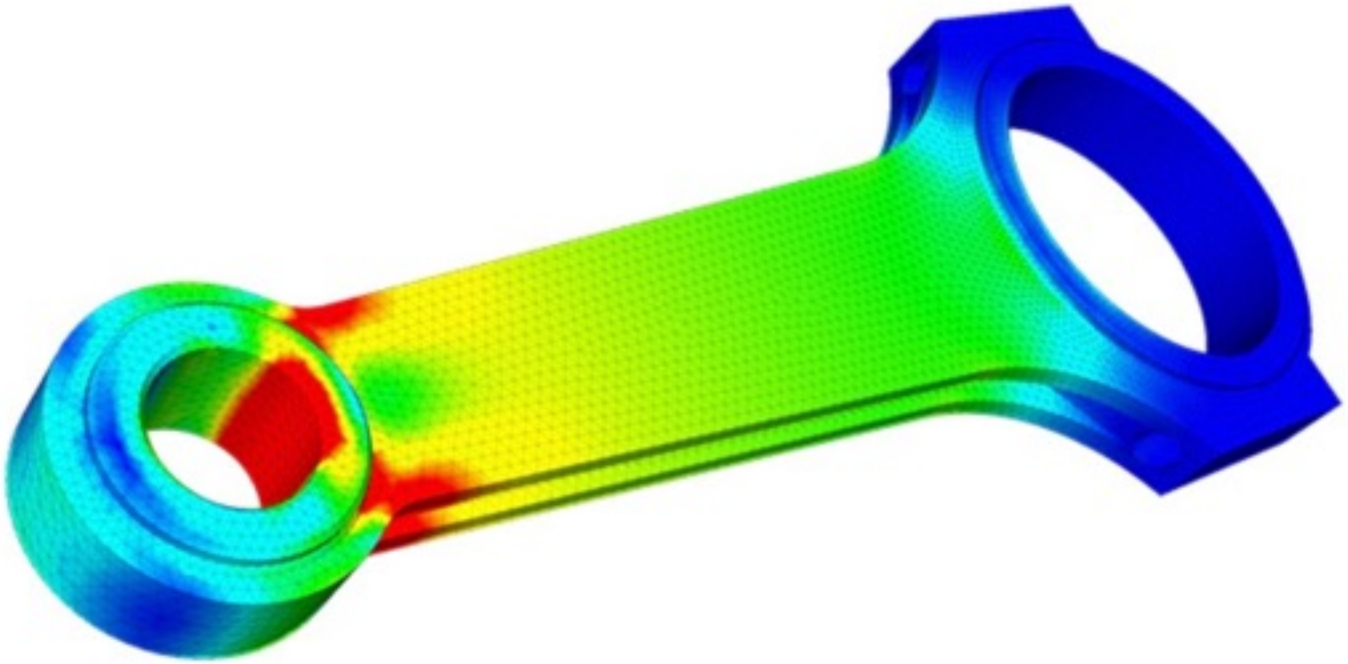
Material Engineering





$$f(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{X}}, R) = J(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{X}}) - \sigma_y - R$$

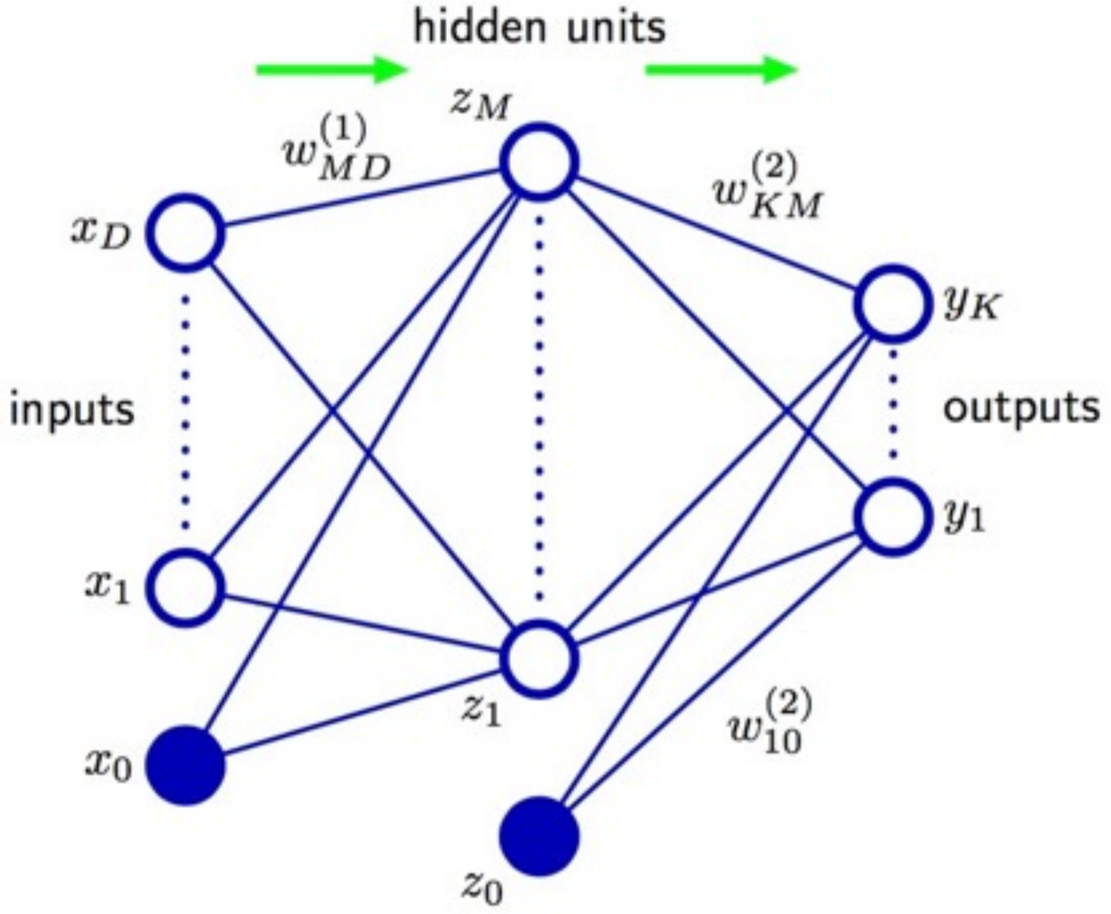
$$R = Q(1 - \exp(-bp))$$



→ Life time prediction?

Machine Learning

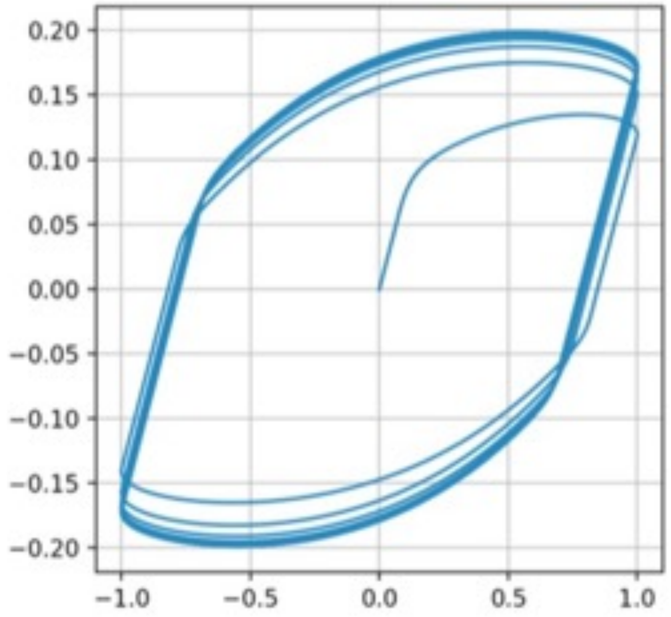
- Regression
  - Model: y = f(x,w;θ). Given the parameter values w and hyperparameters Theta, predictions can be made for any input x (known)
  - Measurements: y_i at x_i
  - Find w, theta

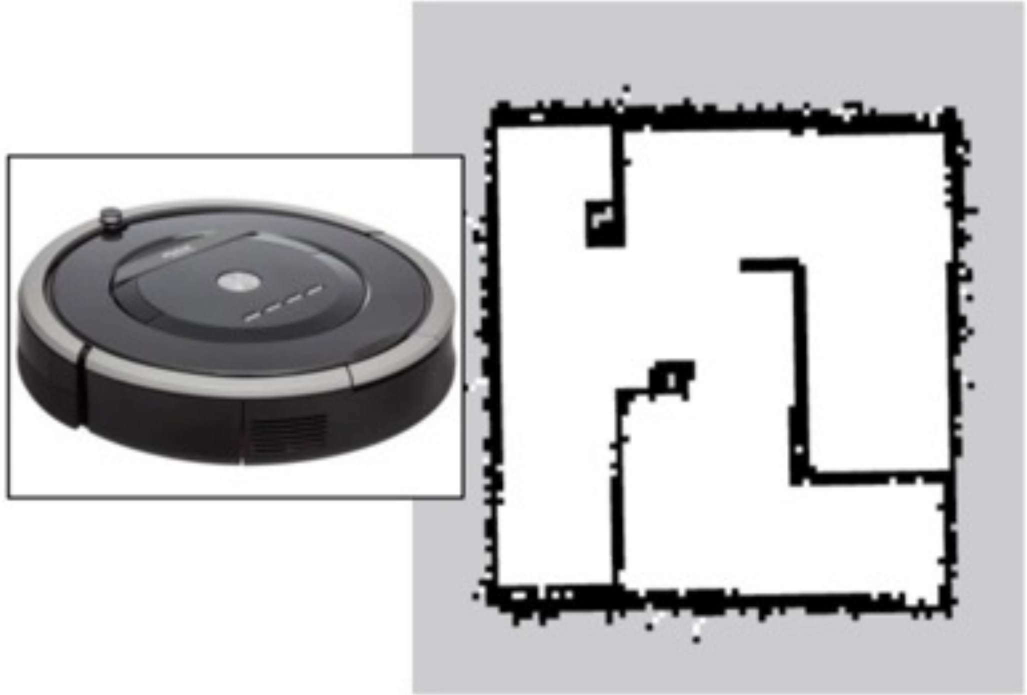


$$y_k(\mathbf{x}, \mathbf{w}) = \sigma\left(\sum_{j=1}^{M} w_{kj}^{(2)} h\left(\sum_{i=1}^{D} w_{ji}^{(1)} x_i + w_{j0}^{(1)}\right) + w_{k0}^{(2)}\right)$$

Generalities

- IPs have small or large numbers of unknowns (e.g. fields, shapes).





→ More unknown needs more data (or more knowledge)

- Main difficulties
  - Highly nonlinear in general (even when the forward model is linear)
  - No solution (wrong model) $\mathbf{d}(\mathbf{p}) \neq \mathbf{d}_{true}$
  - Many solutions (not enough data? Not enough prior knowledge?)
  - Bad conditioning -> perturbation leads to completely different solutions

$$\mathbf{d}_{obs} = \mathbf{d}_{true} + \epsilon$$

Typical example

$$\mathbf{G} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{G}^{-1} = \begin{bmatrix} 25 & 41 & 10 & -6 \\ -41 & 68 & -17 & 10 \\ 10 & -17 & 5 & -3 \\ -6 & 10 & -3 & 2 \end{bmatrix}$$

Effect of perturbations of $\mathbf{G}$ or $\mathbf{d}$ on the solution of $\mathbf{Gp} = \mathbf{d}$

$$\mathbf{d} = [32 \;\; 23 \;\; 33 \;\; 31]^{\mathsf{T}} \Longrightarrow \mathbf{p} = [1 \;\; 1 \;\; 1 \;\; 1]^{\mathsf{T}}$$

$$\delta\mathbf{d} = [0.1 \;\; -0.1 \;\; 0.1 \;\; -0.1]^{\mathsf{T}} \Longrightarrow \mathbf{p} = [9.2 \;\; -12.6 \;\; 4.5 \;\; -1.1]^{\mathsf{T}}$$

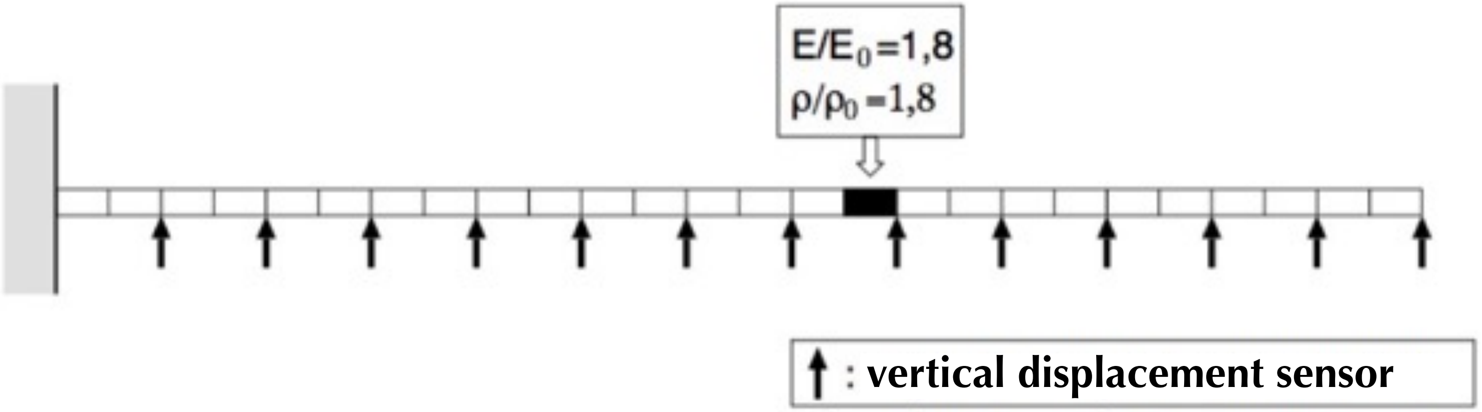
$$\delta\mathbf{G}_{23} = 0.1 \Longrightarrow \mathbf{p} \approx [-4.86 \;\; -10.7 \;\; -1.43 \;\; -2.43]^{\mathsf{T}}$$

Eigenvalues and conditioning of $\mathbf{G}$

$$\mathbf{\Lambda} \approx \mathrm{Diag}[\, 30.29 \;\; 3.858 \;\; 0.8431 \;\; 0.01015 \,], \qquad \boxed{\mathrm{cond}(\mathbf{G}) \approx 2.98\, 10^3}$$

Very large conditioning for a 4x4 matrix!

Structure vibrations



Reconstruct $EI(x), \rho(x)$ from data $\omega_i (1 \leq i \leq n_f), u_{ij} (1 \leq j \leq n_c)$

- 52 elements, 26 macro-elements
- $n_f = 10$ measured modes, $n_c = 13$ sensors
- macro-element 16: $EI(x) = 1.8 EI_0, \rho(x) = 1.8 \rho_0$
- other macro-elements: $EI(x) = EI_0, \rho(x) = \rho_0$

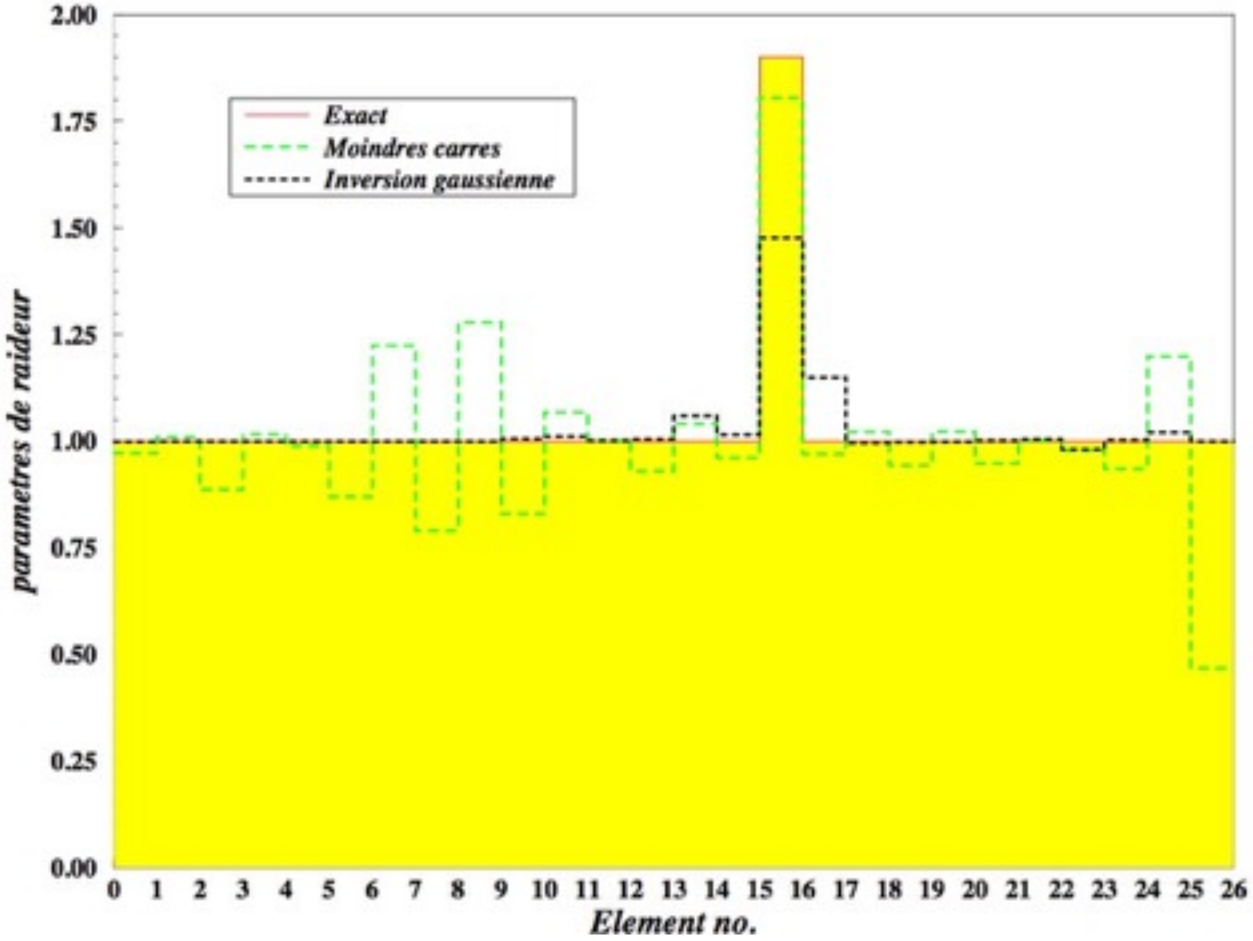
Minimization of a quadratic function

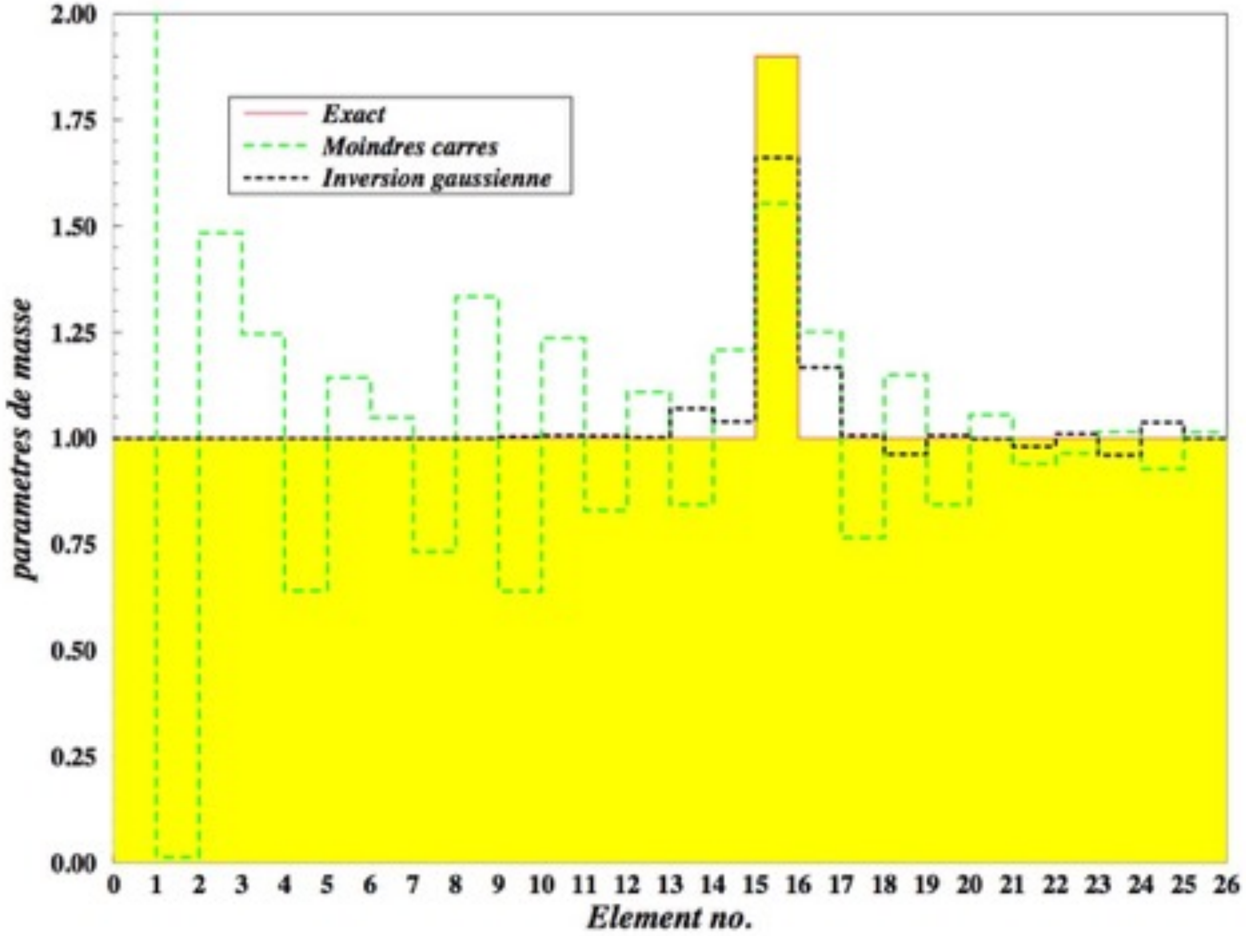
$$\boxed{\min_{EI,\rho} \sum_{i=1}^{n_f} \left([f_i^{mes} - f_i(EI,\rho)]^2 + \sum_{j=1}^{n_c} [u_{ij}^{mes} - u_{ij}(EI,\rho)]^2 \right)}$$

Synthetic data: noisy data = exact data x (1+r)

uniform r.v. ($\langle r \rangle = 0, \sigma = 10^{-3}$)

Structure vibrations





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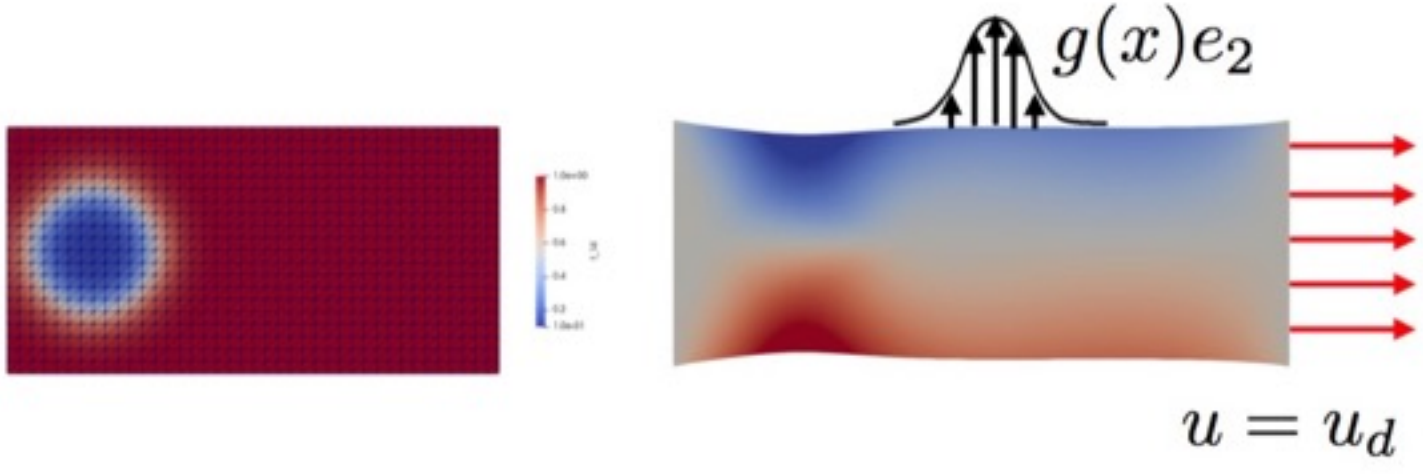
2D Problem of Interest

- Parametrised forward elasticity problem

$$\nabla \cdot (C(\boldsymbol{\mu})\nabla_s u) = 0$$

$$u = u_d \quad \text{in}\, \Gamma_r$$

$$u = 0 \quad \text{in}\, \Gamma_l$$



- Parameters of diffusion field

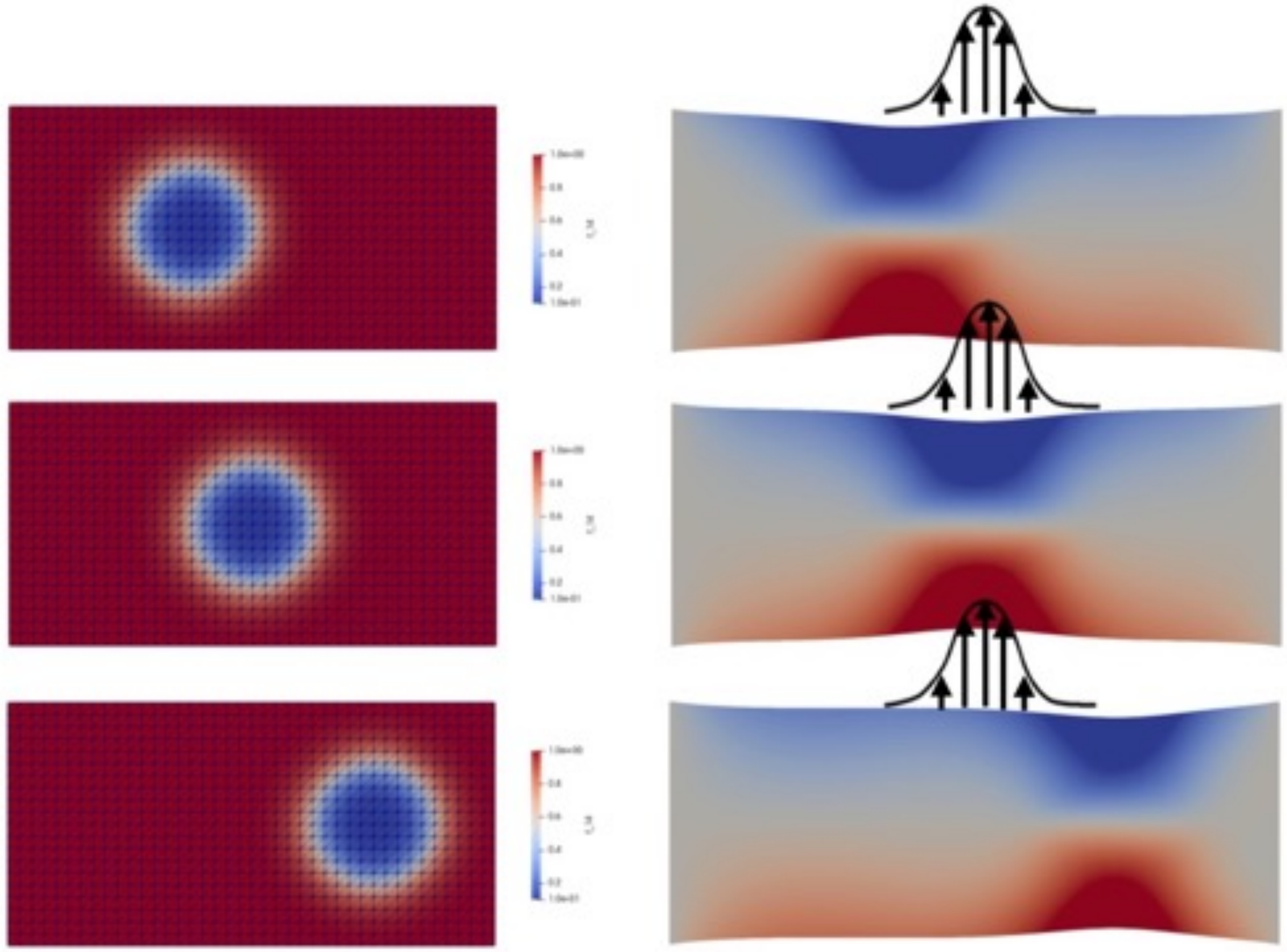
$$\boldsymbol{\mu} = (\boldsymbol{\mu}_1 \;\; \boldsymbol{\mu}_2)^T$$

Radius

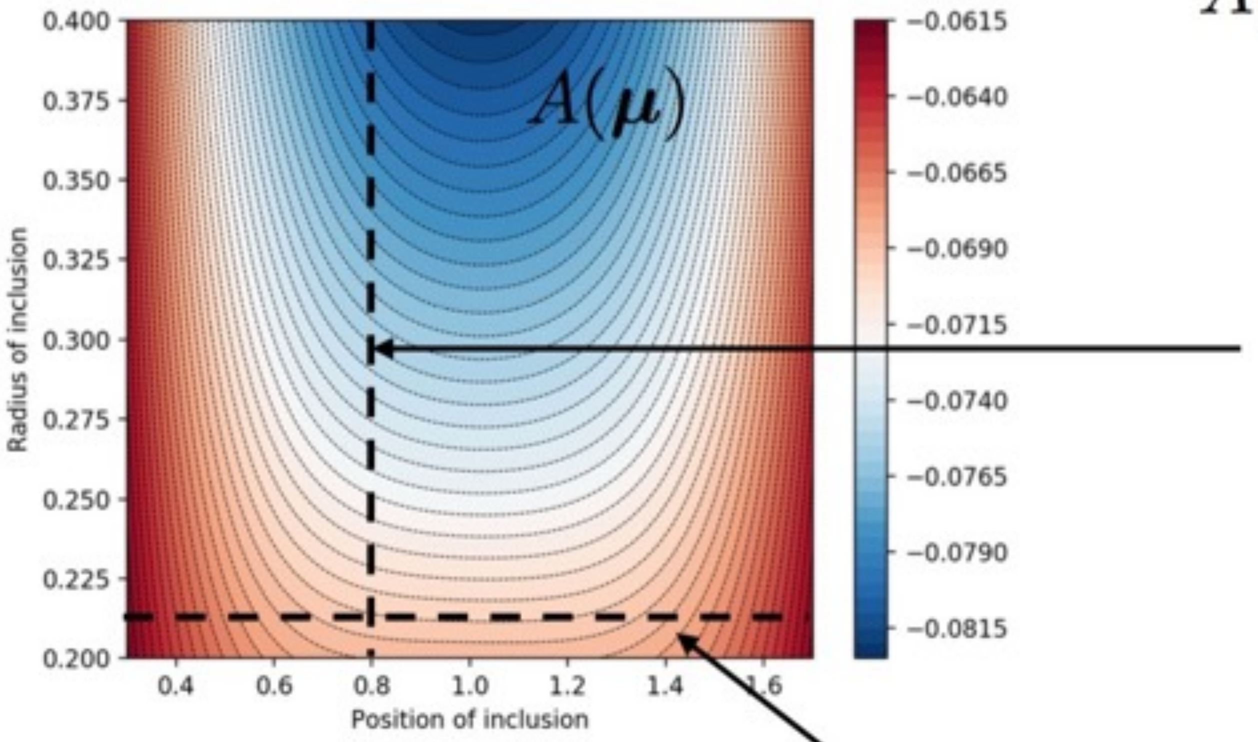
Position along e_1

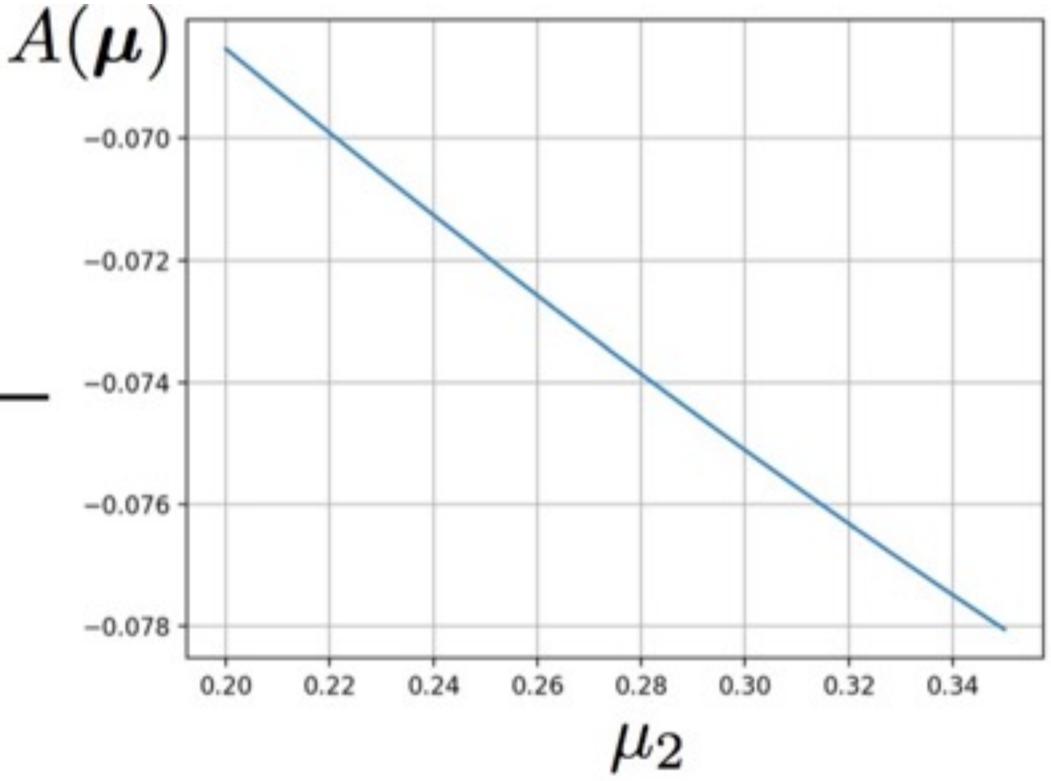
- Measurement: vertical displacement at mid-span

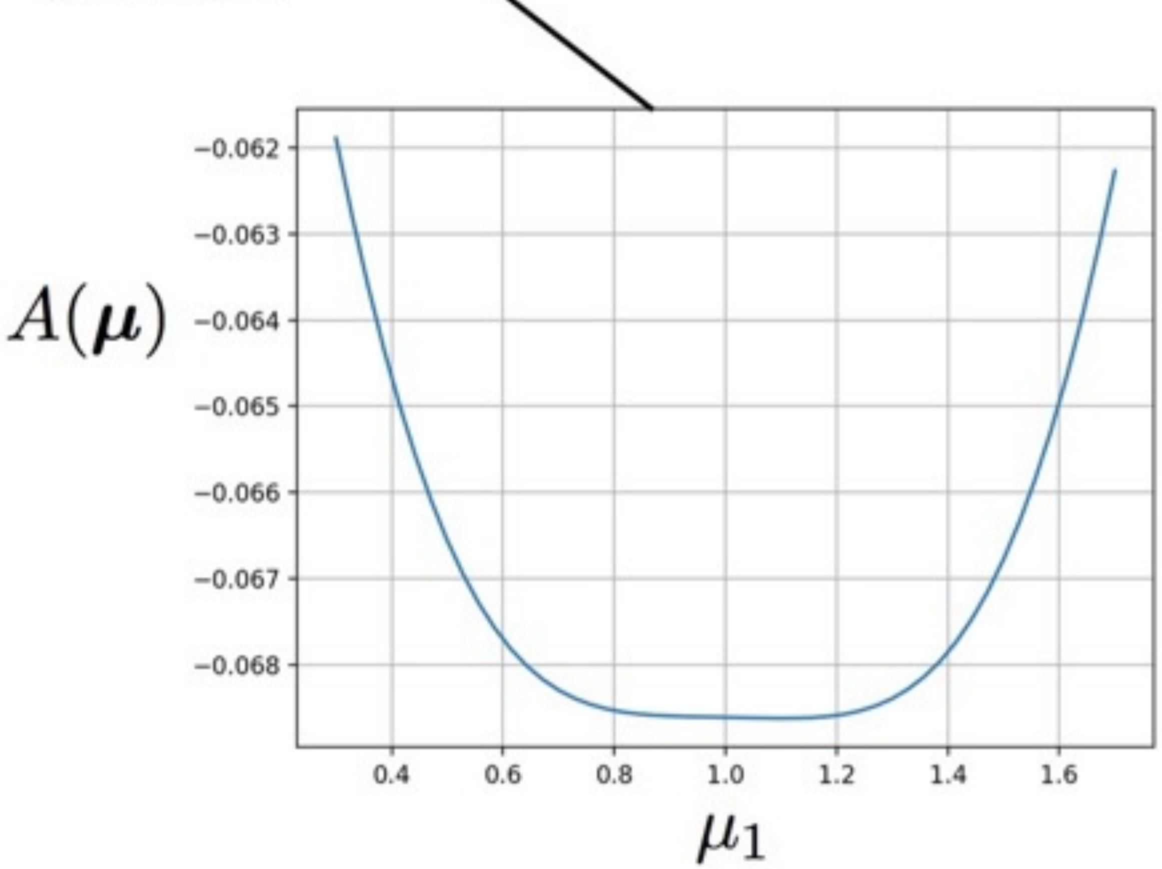
$$A(\boldsymbol{\mu}) := \int_{\Gamma^+} (u(\boldsymbol{\mu}) \cdot g(x)e_2)\, dx$$



Observations vs Parameters







Multiple or no solution at all!

This is the forward map, can only be exhaustively populated for a small number of parameters (less than 5)

Optimization formalism

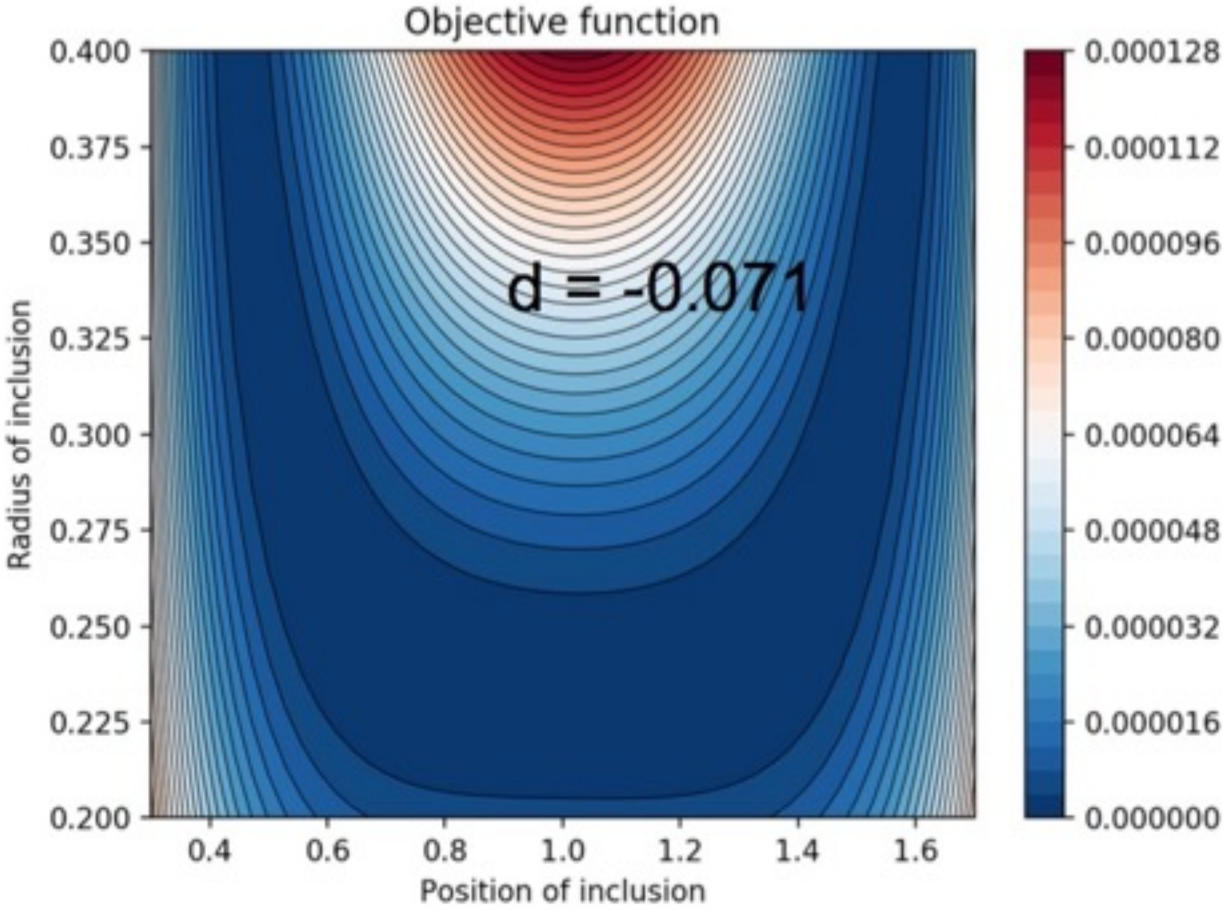
Inverse problem: Given $\mathbf{d} \in \mathbb{R}^m$, find $\boldsymbol{\mu}^\star \in \mathbb{R}^n$ such that $\mathbf{A}(\boldsymbol{\mu}) \cancel{=} \mathbf{d}$

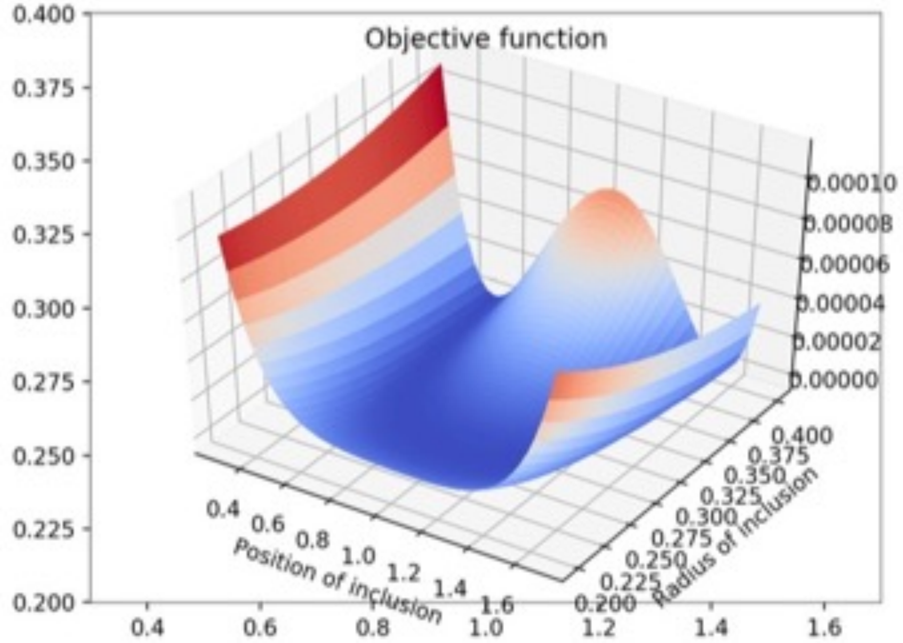
$$\boxed{\boldsymbol{\mu}^\star = \arg\min_{\boldsymbol{\mu}} \frac{1}{2}\|\mathbf{d} - \mathbf{A}(\boldsymbol{\mu})\|^2_{\mathbf{X}} = \arg\min_{\boldsymbol{\mu}} E(\boldsymbol{\mu})}$$

$$\|\,.\,\|^2_{\mathbf{X}} = .^T\mathbf{X}\,.$$

→ Has always at least one solution

→ May have an infinity of solutions





☐ Computation of a solution?

☐ Definition and computation of the “right” solution amongst all those of in this valley?

Gradient-based Algorithms

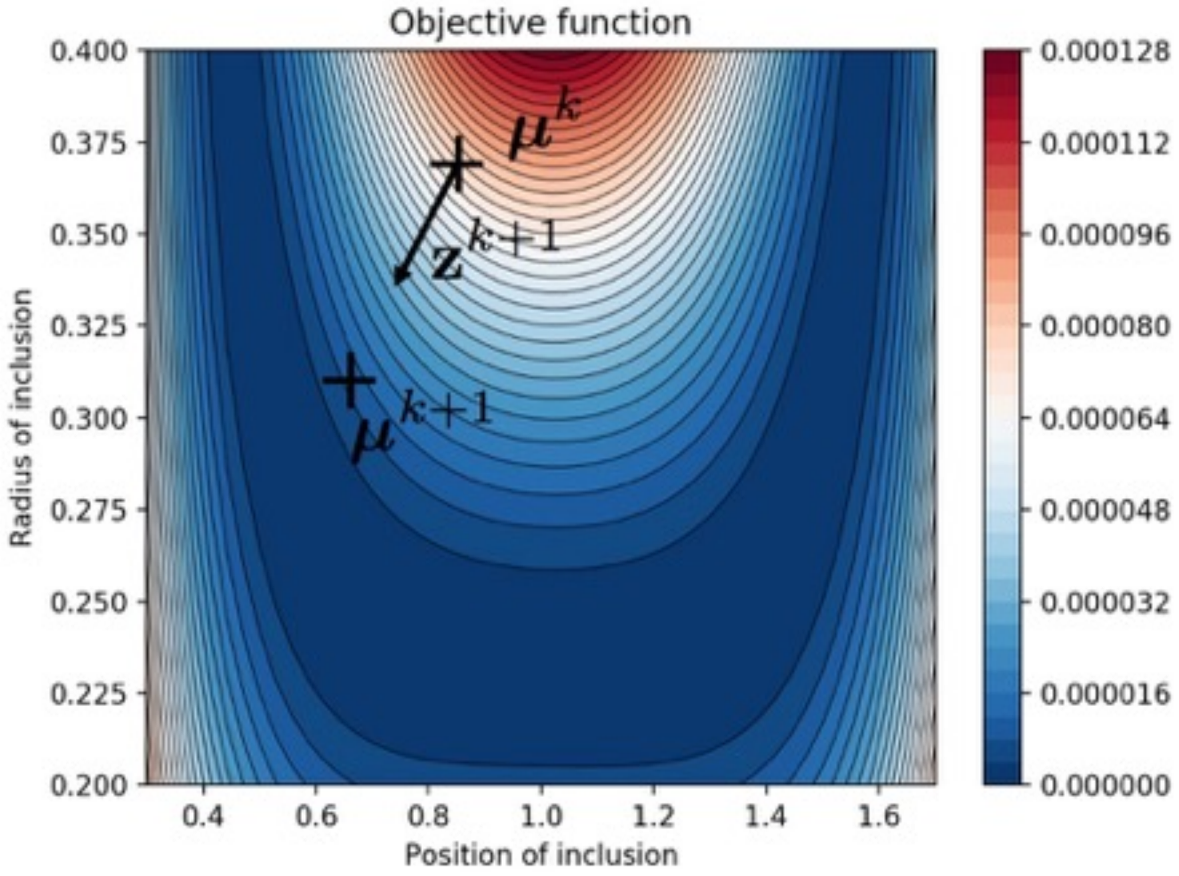
$$\boldsymbol{\mu}^\star = \arg\min_{\boldsymbol{\mu}} \frac{1}{2}\|\mathbf{d} - \mathbf{A}(\boldsymbol{\mu})\|^2_{\mathbf{X}} = \arg\min_{\boldsymbol{\mu}} E(\boldsymbol{\mu})$$

$$\Delta\boldsymbol{\mu}^{k+1} = \gamma^{k+1}\mathbf{z}^{k+1}$$

increment

Search radius

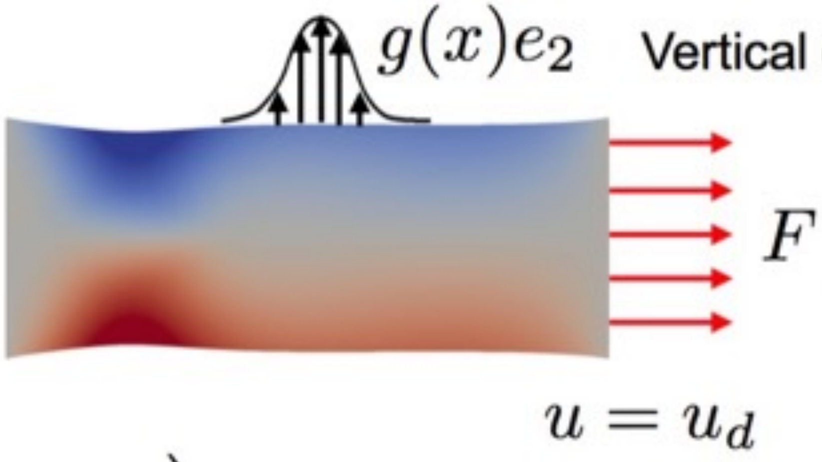
Search direction (towards minimum, for instance negative gradient)



- Steepest descent (finite difference, adjoint method,…)
- Newton's method (Hessian computation, Gauss-Newton, Levenberg-Marquardt,…)
- Derivative-free solver (Nelder-Mead (downhill simplex), genetic algorithms,…)

Regularization: less dofs

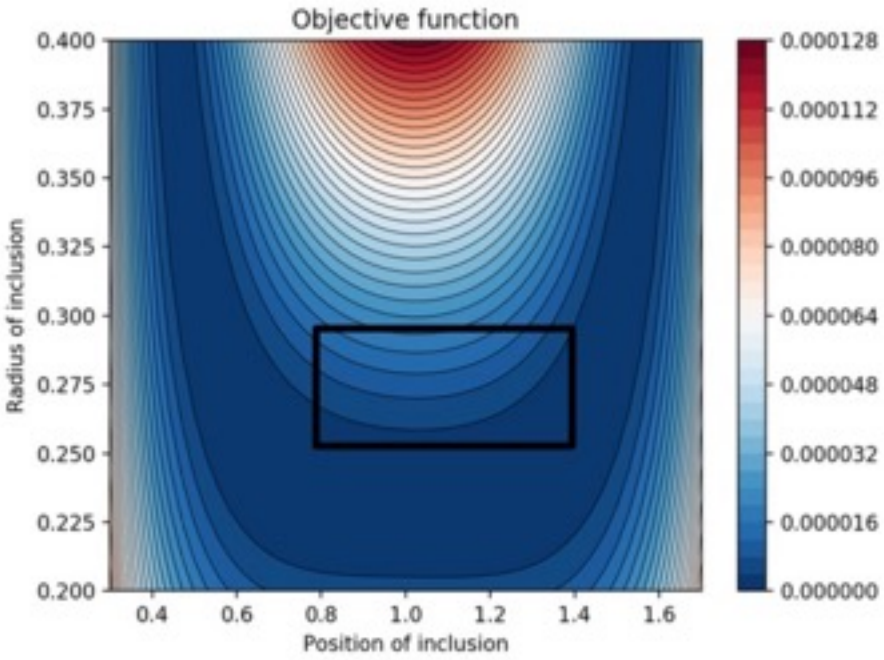
More data:

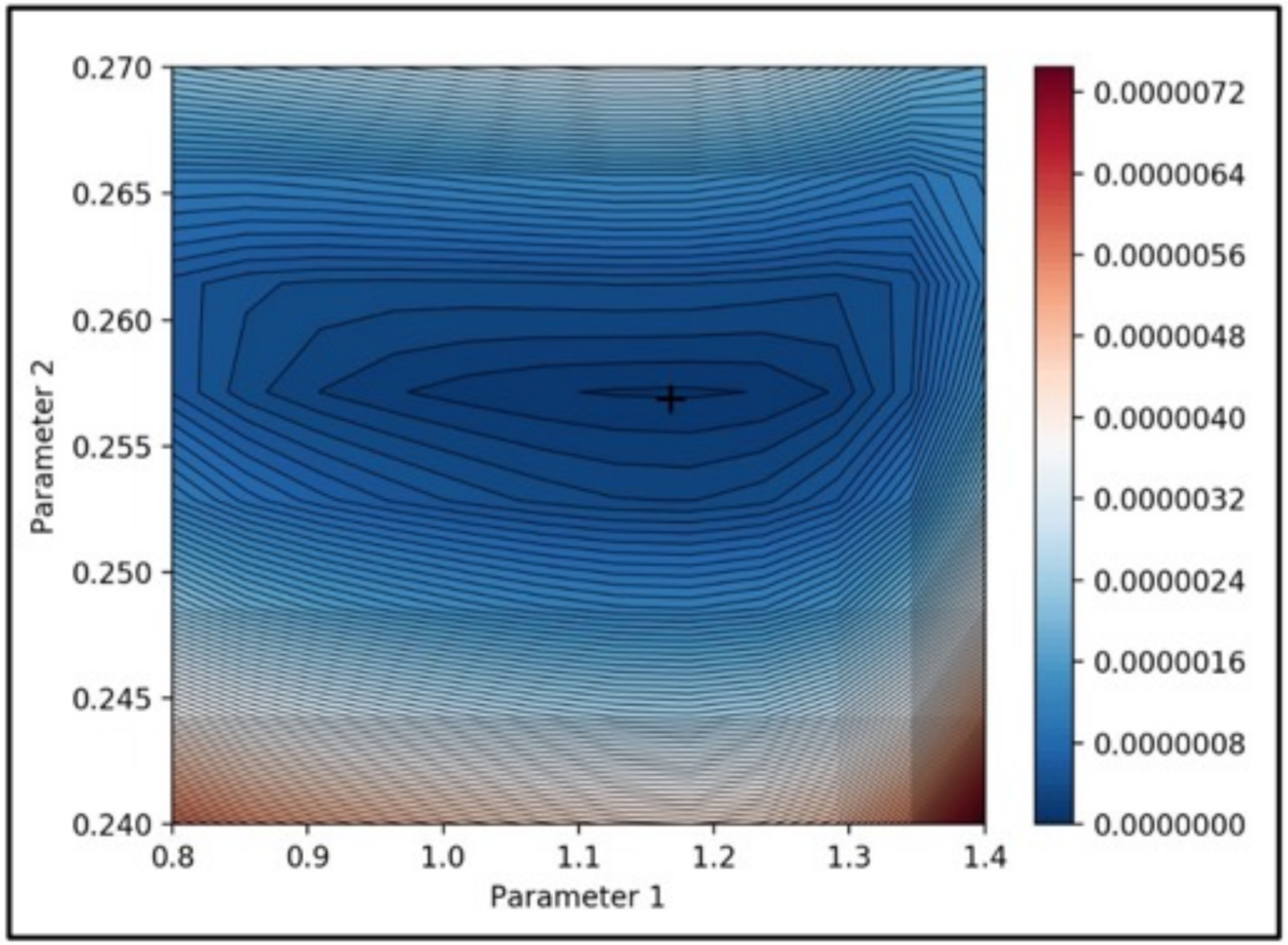


Vertical displacement

Reaction force now measured (sensitive to size of inclusion)

$$\mathbf{A}(\boldsymbol{\mu}) = \begin{pmatrix} \int_{\Gamma^+} (u(\boldsymbol{\mu}) \cdot g(x)e_2)\,dx \\ \int_{\Gamma_r} (C(\boldsymbol{\mu})\nabla_s u(\boldsymbol{\mu}) \cdot e_1)\,dx \end{pmatrix}$$





More knowledge: remove parameters or provide tighter bounds for parameter domain

Regularization: soft prior belief

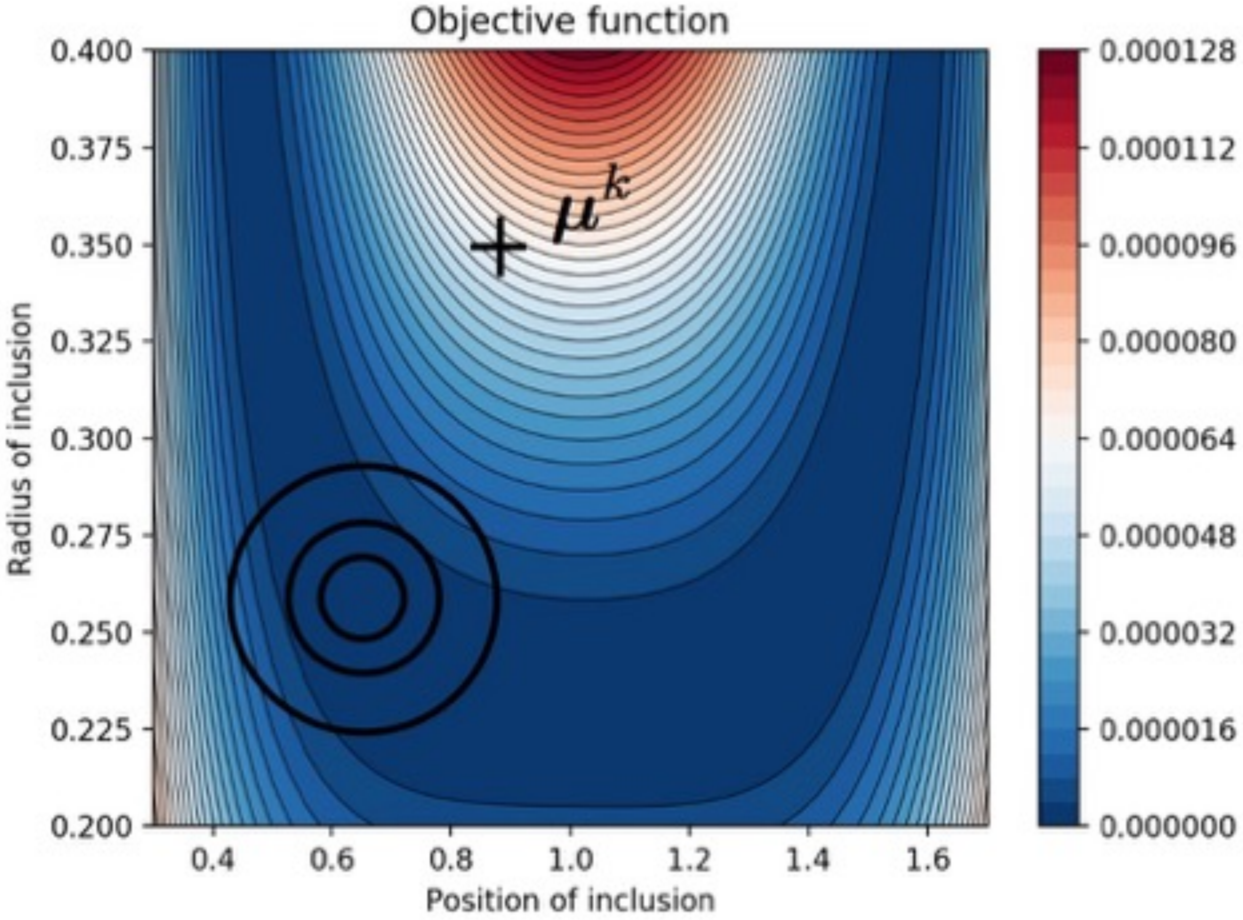
[Tikhonov 77]

$$\boldsymbol{\mu}^\star = \arg\min_{\boldsymbol{\mu}} \left(\frac{1}{2}\|\mathbf{d} - \mathbf{A}(\boldsymbol{\mu})\|^2 + \frac{\alpha}{2}\|\boldsymbol{\mu} - \boldsymbol{\mu}_0\|^2 \right) = \arg\min_{\boldsymbol{\mu}} \left(\tilde{E}(\boldsymbol{\mu}) + E_0(\boldsymbol{\mu}) \right)$$

$\boldsymbol{\mu}$ required to be close to $\boldsymbol{\mu}_0$

α is the strength of the link

→ Penalises large componentwise deviations from $\boldsymbol{\mu}_0$



$$E(\boldsymbol{\mu}) \approx E(\boldsymbol{\mu}^\star) + \frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}^\star)^T \cdot (\tilde{\mathbf{H}}_{|\boldsymbol{\mu}^\star} + \alpha\mathbf{I}) \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}^\star)$$

→ Introduces bias if strong regularisation and objectif is not minimum at belief

→ Helps when number of parameters is large, also smoothness & physics

→ Alternative regularisers: smoothness and/or penalty away from physics

modified CRE (mCRE) Framework

[Ladevèze *et al.* 94, Chouaki 96, Deraemaeker 04]

Framework based on reliability of information

$$\mathcal{E}^2_{mCRE}(\hat{\mathbf{u}},\hat{\sigma};\mathbf{p}) = \underset{\text{modeling error term (CRE)}}{\boxed{\mathcal{E}^2_{CRE}(\hat{\mathbf{u}},\hat{\sigma};\mathbf{p})}} + \frac{\alpha}{2}\underset{\text{distance to measurements (displacements, forces,…)}}{\boxed{(\mathbf{d}(\hat{\mathbf{u}})-\mathbf{d}_{obs})^T\mathbb{G}^{-1}_{obs}(\mathbf{d}(\hat{\mathbf{u}})-\mathbf{d}_{obs})}}$$

The problem is split in : - reliable part (BC, equilibrium,…) ⟶ admissible solution

- unreliable part (material behavior, sensor values, BC…)

→ enforces **reliable theor./exp. info**: admissibility (regularization from physics)

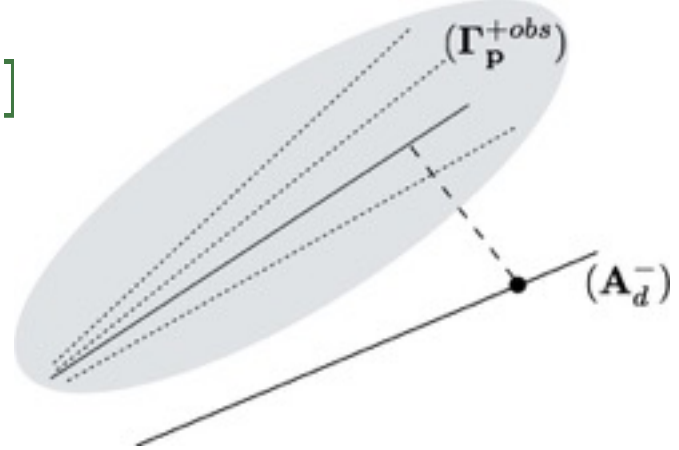
→ hybrid energy-based formulation

→ high convexity

→ robust with noisy/corrupted data [Allix 05, Feissel & Allix 07]

→ explicit model error (localization + correction)

$$\boxed{\mathbf{p}_{sol} = argmin_{\mathbf{p}\in\mathcal{P}}\left[\min_{(\hat{\mathbf{u}},\hat{\sigma})\in(\mathbf{A_d}^-)}\mathcal{E}^2_{mCRE}(\hat{\mathbf{u}},\hat{\sigma};\mathbf{p})\right]}$$

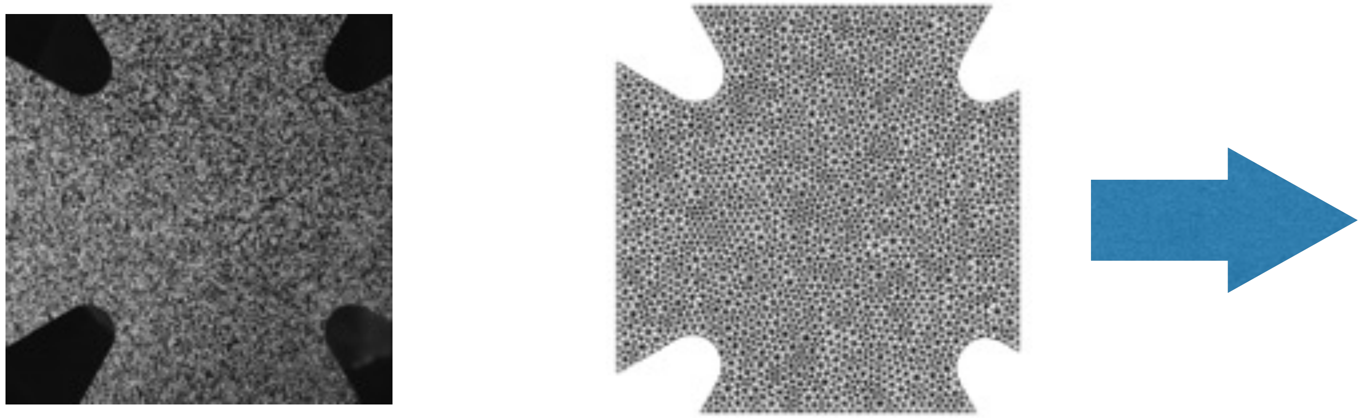


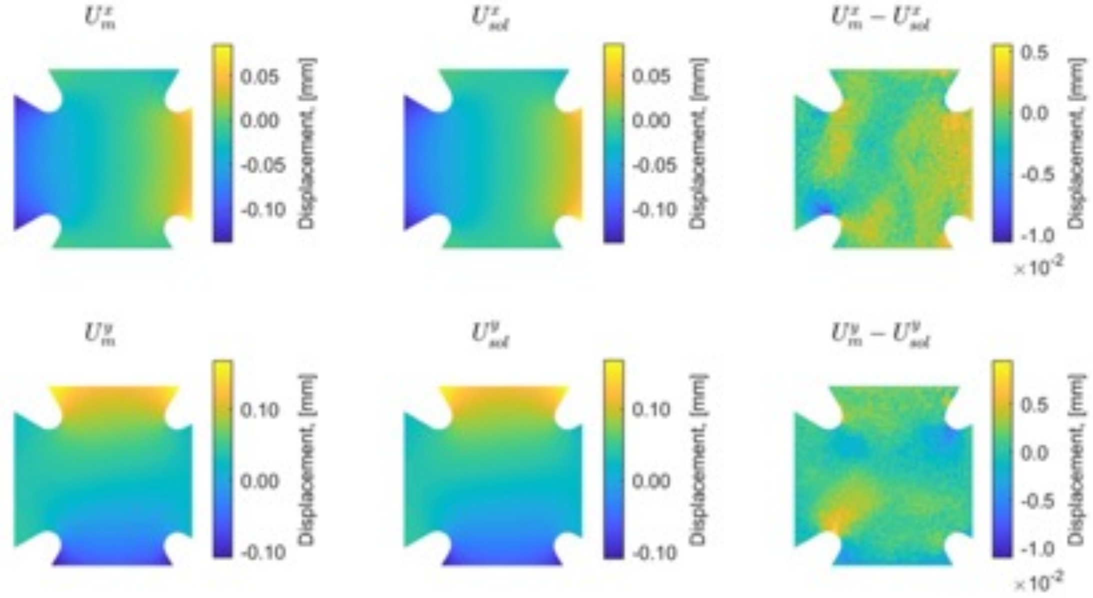
Link with Bayesian: $\pi(\mathbf{d}_{obs}|\mathbf{p}) = C_1.e^{-\frac{1}{2}(\mathbf{d}_{obs}-\mathbf{d}(\hat{\mathbf{u}}(\mathbf{d})))^T\Sigma^{-1}_{obs}(\mathbf{d}_{obs}-\mathbf{d}(\hat{\mathbf{u}}(\mathbf{p})))}.e^{-\underset{\uparrow\,\pi_{mod}}{\frac{\mathcal{E}^2_{CRE}(\hat{\mathbf{u}},\hat{\sigma};\mathbf{p})}{\alpha}}}$

Example 1: Full-field Measurements

[N. Nguyen & Chamoin 22]

$f(\mathbf{x}) \longrightarrow g(\mathbf{x})$ *(reference/deformed)*



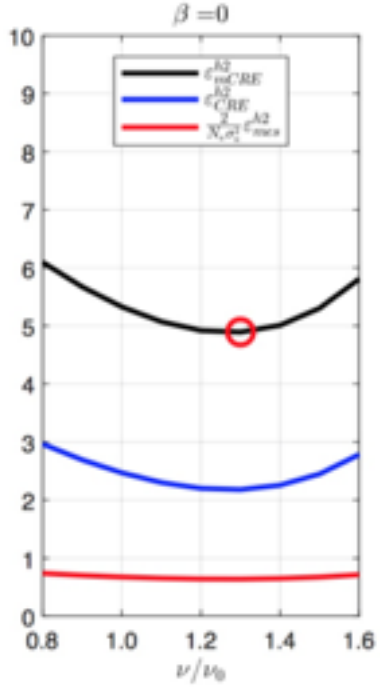


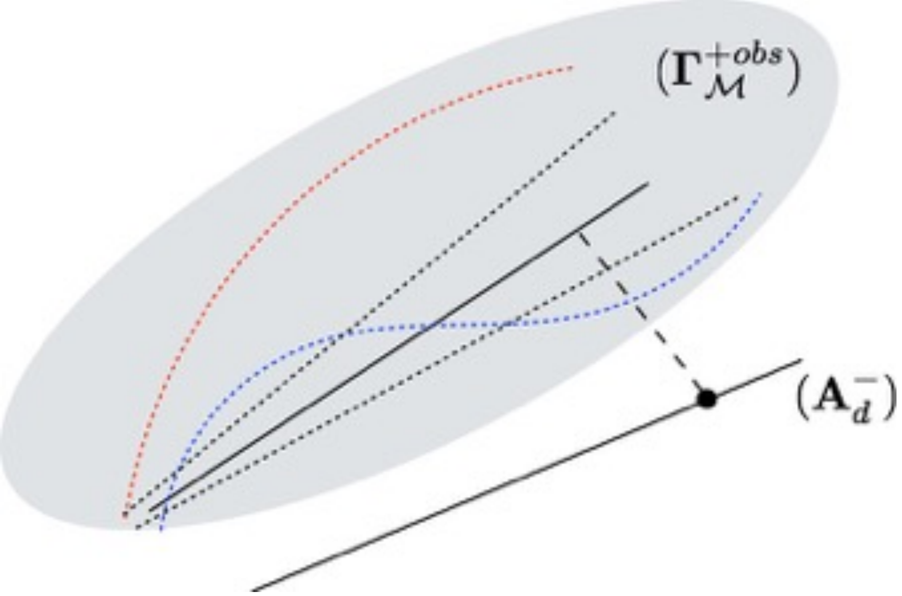
mIDIC (integrated version):

$$\mathcal{E}^2_{mIDIC}(\hat{\mathbf{U}},\hat{\mathbf{V}},\mathbf{p}) = \boxed{\frac{1}{2}(\hat{\mathbf{U}}-\hat{\mathbf{V}})^T\mathbb{K}(\mathbf{p})(\hat{\mathbf{U}}-\hat{\mathbf{V}})} + \frac{\alpha}{2}\boxed{\frac{1}{2\gamma_f^2 N_{pix}}\sum_{\mathbf{x}\in ROI}\Big(f(\mathbf{x})-g(\mathbf{x}+\mathbb{N}(\mathbf{x})\hat{\mathbf{U}})\Big)^2}$$

CRE term
(modeling bias, adaptivity)

correlation residual
(gray level conservation)





drive the complexity of the constitutive law (or learn it!!!)

Example 2: Viscoplasticity

[Marchand *et al.* 15]

- extension to NL constitutive models using dual convex thermodynamics potentials

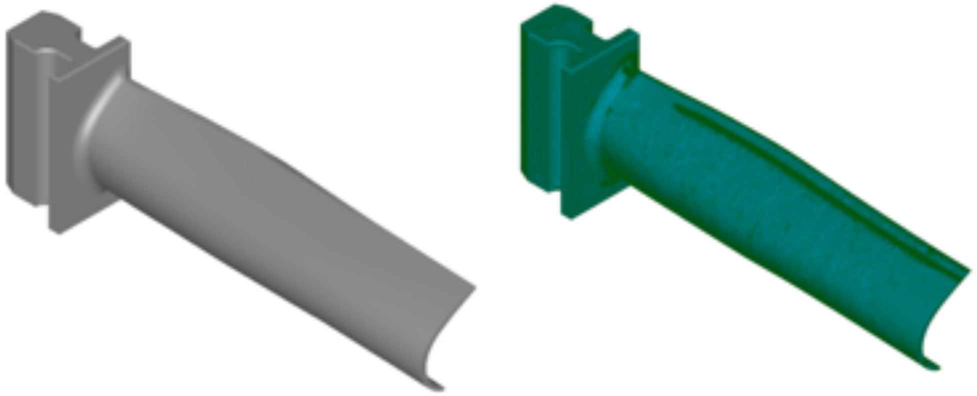
→ CRE measure from Legendre-Fenchel residuals (sym. Bergman divergence)

$$\boxed{\mathcal{E}^2_{CRE|t} = \int_\Omega \eta_\psi(\hat{\mathbf{e}}_e, \hat{\mathbf{s}}) + \int_0^t \int_\Omega \eta_\varphi(\dot{\hat{\mathbf{e}}}_p, \hat{\mathbf{s}})}$$

with

$$\left| \begin{array}{l} \eta_\psi(\hat{\mathbf{e}}_e, \hat{\mathbf{s}}) = \psi(\hat{\mathbf{e}}_e) + \psi^*(\hat{\mathbf{s}}) - \hat{\mathbf{s}} \cdot \hat{\mathbf{e}}_e \\ \eta_\varphi(\dot{\hat{\mathbf{e}}}_p, \hat{\mathbf{s}}) = \varphi(\dot{\hat{\mathbf{e}}}_p) + \varphi^*(\hat{\mathbf{s}}) - \hat{\mathbf{s}} \cdot \dot{\hat{\mathbf{e}}}_p \end{array} \right.$$

Rotating turbine blade (viscoplasticity)

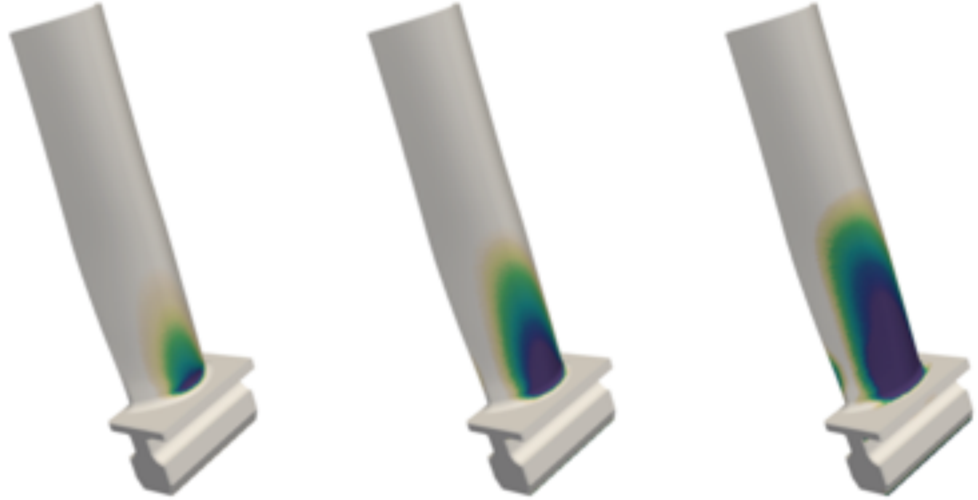


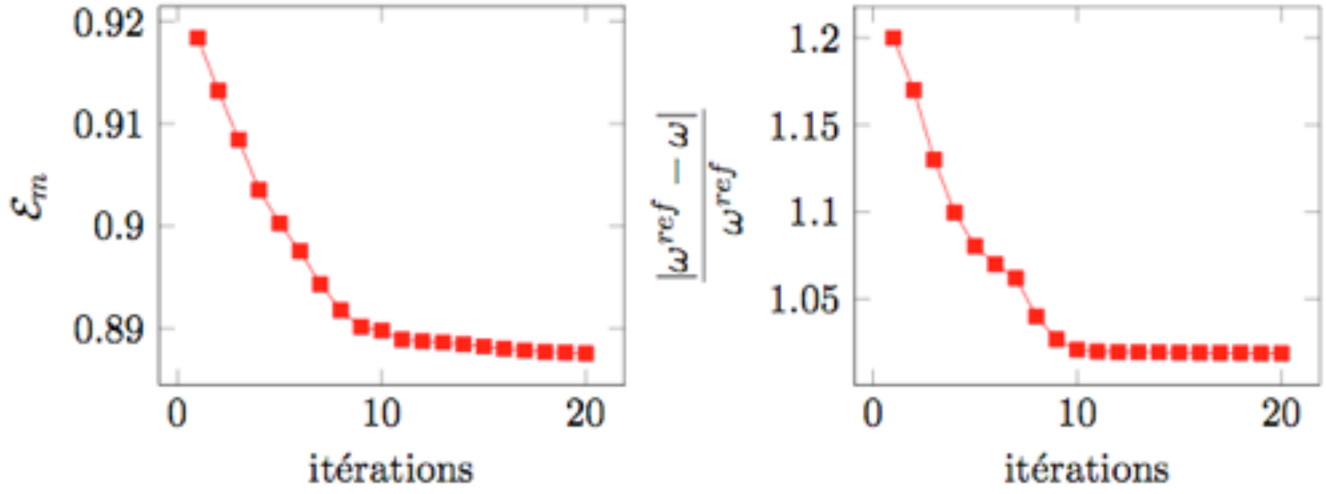
Prandlt-Reuss model
with isotropic hardening

$$\begin{aligned} \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{vp}, p) &= \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp}) : \boldsymbol{\mathcal{C}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp}) + g(p) \\ \varphi^\star(\boldsymbol{\sigma}, R) &= \frac{k_v}{n_v + 1} \left\langle \|\boldsymbol{\sigma}^D\| - (R + R_0) \right\rangle_+^{n_v+1} \end{aligned}$$

8 sensors on extrado +
8 sensors on intrado

$$\begin{aligned} \psi^\star(\boldsymbol{\sigma}, R) &= \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\mathcal{C}}^{-1} : \boldsymbol{\sigma} + g^\star(R) \\ \varphi(\dot{\boldsymbol{\varepsilon}}^{vp}, -\dot{p}) &= R_0 \|\dot{\boldsymbol{\varepsilon}}^{vp}\| + \frac{1}{(n_v + 1) k_v^{n_v}} \|\dot{\boldsymbol{\varepsilon}}^{vp}\|^{n_v+1} \end{aligned}$$

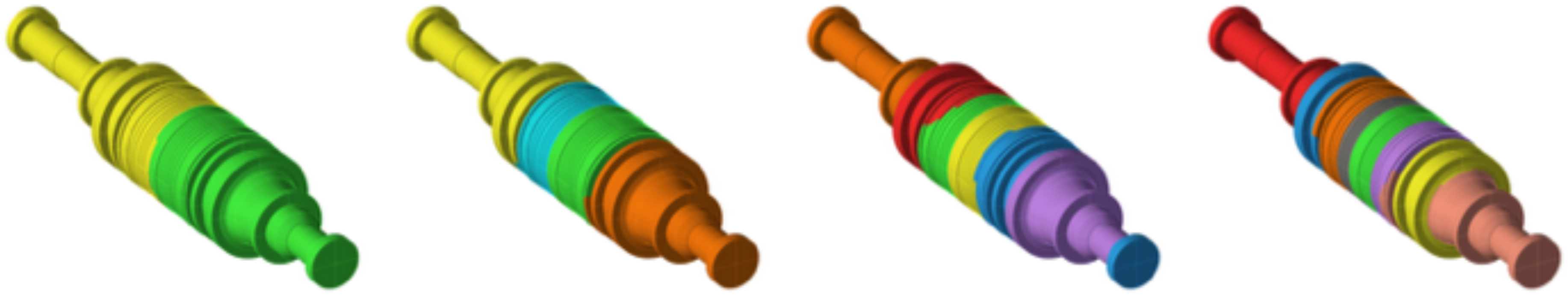


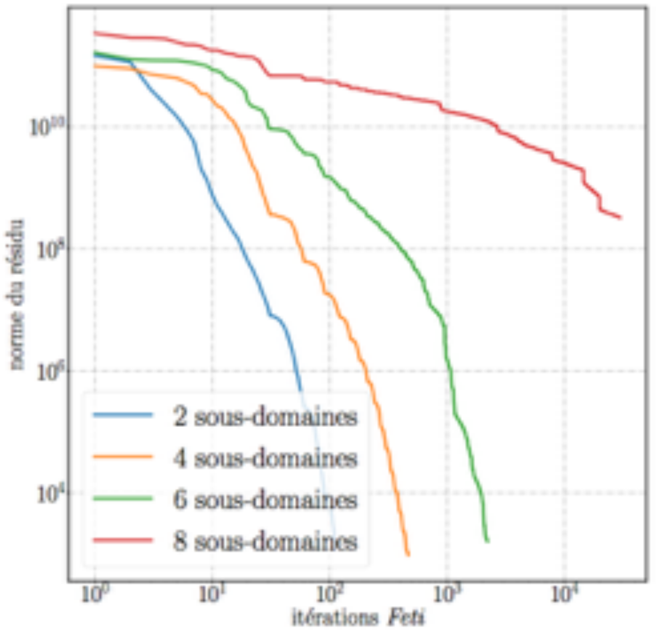


Recent Applications

- **Dynamics with domain Decomposition + ROM (implementation in Code Aster)**

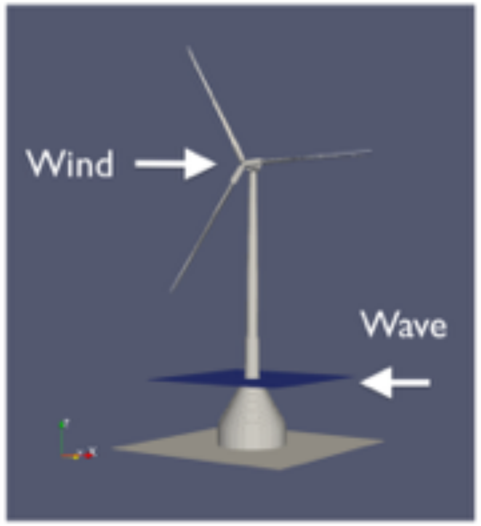
  [PhD Z. Samir 2019-2022]

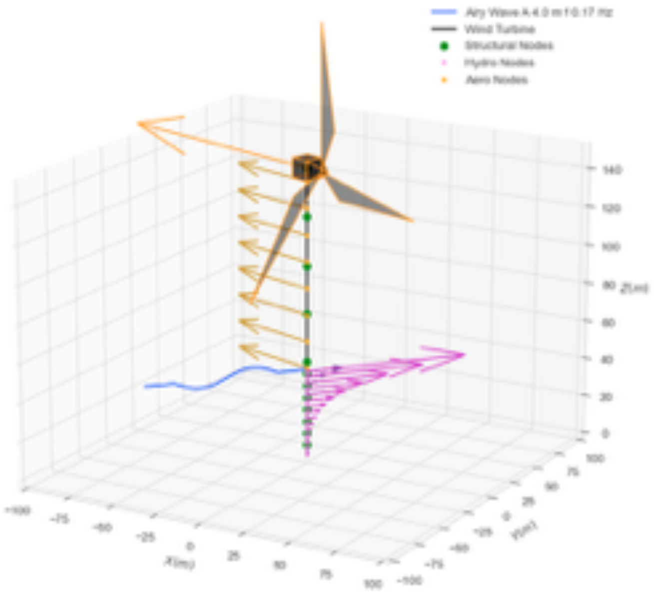


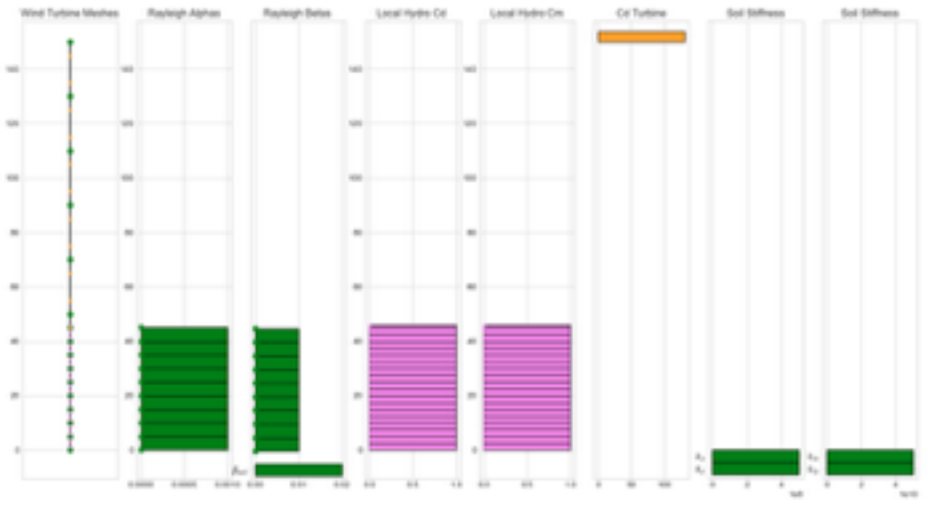


- **Fluid-Structure interaction (several interface constitutive laws)**

  [PhD A. Roussel 2020-2023]







Outline

1. Introduction to inverse problems
2. Deterministic inverse problems
3. Stochastic inverse problems
4. Sequential data assimilation
5. Some recent research applications

Bayesian Inference

x parameters

y observations (impacted by noise) → considered as random variables

$$\boxed{p(x|y) = \frac{1}{p(y)} p(y|x)p(x)}$$

→ how the state of knowledge of x is changed as a result of making an observation which yields y

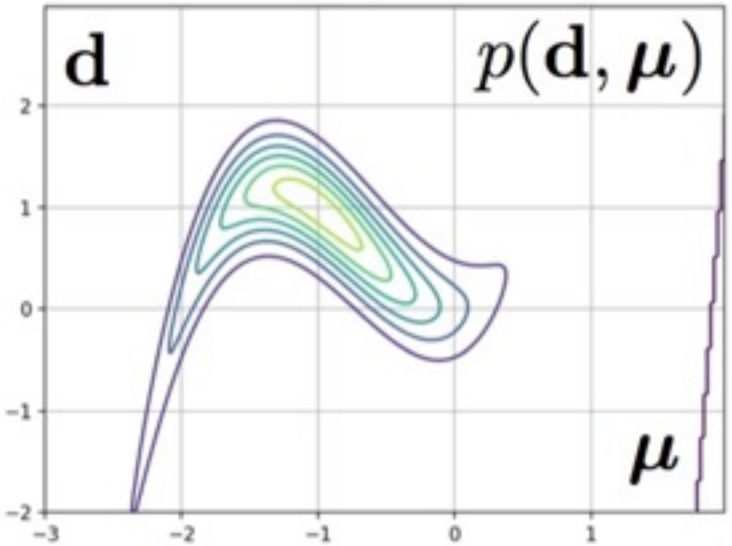
$p(x)$: what we know about x before making the observation (prior probability)

$p(x|y)$: what we know about x after making the observation (posterior probability)

$p(y|x)$: forward probability (likelihood function of x for a fixed observation y)

$p(y)$: normalization term

Bayesian Inverse Problem



- Construct joint probability density $p(\mathbf{d}, \boldsymbol{\mu})$ for the observations and the parameters
  Use the chain sum rule $p(\mathbf{d}, \boldsymbol{\mu}) = p(\mathbf{d}|\boldsymbol{\mu})p(\boldsymbol{\mu})$
  - $p(\boldsymbol{\mu})$ is the prior belief about our parameters. Can be made arbitrarily flat (large variance) if not much knowledge is available. Can use sequential knowledge aquisition
  - Likelihood $p(\mathbf{d}|\boldsymbol{\mu})$ obtained by assuming that data that will be measured are generated by the model $\mathbf{A}(\boldsymbol{\mu})$, and polluted by an additive Gaussian noise

$$\mathbf{d} = \mathbf{A}(\boldsymbol{\mu}) + \boldsymbol{\epsilon} \qquad\qquad p(\mathbf{d}|\boldsymbol{\mu}) = \mathcal{N}(\mathbf{A}(\boldsymbol{\mu}), \boldsymbol{\Sigma})$$

- Use Bayes formula, $p(\boldsymbol{\mu}|\mathbf{d}) = \dfrac{p(\mathbf{d}, \boldsymbol{\mu})}{p(\mathbf{d})}$ where d is now measured

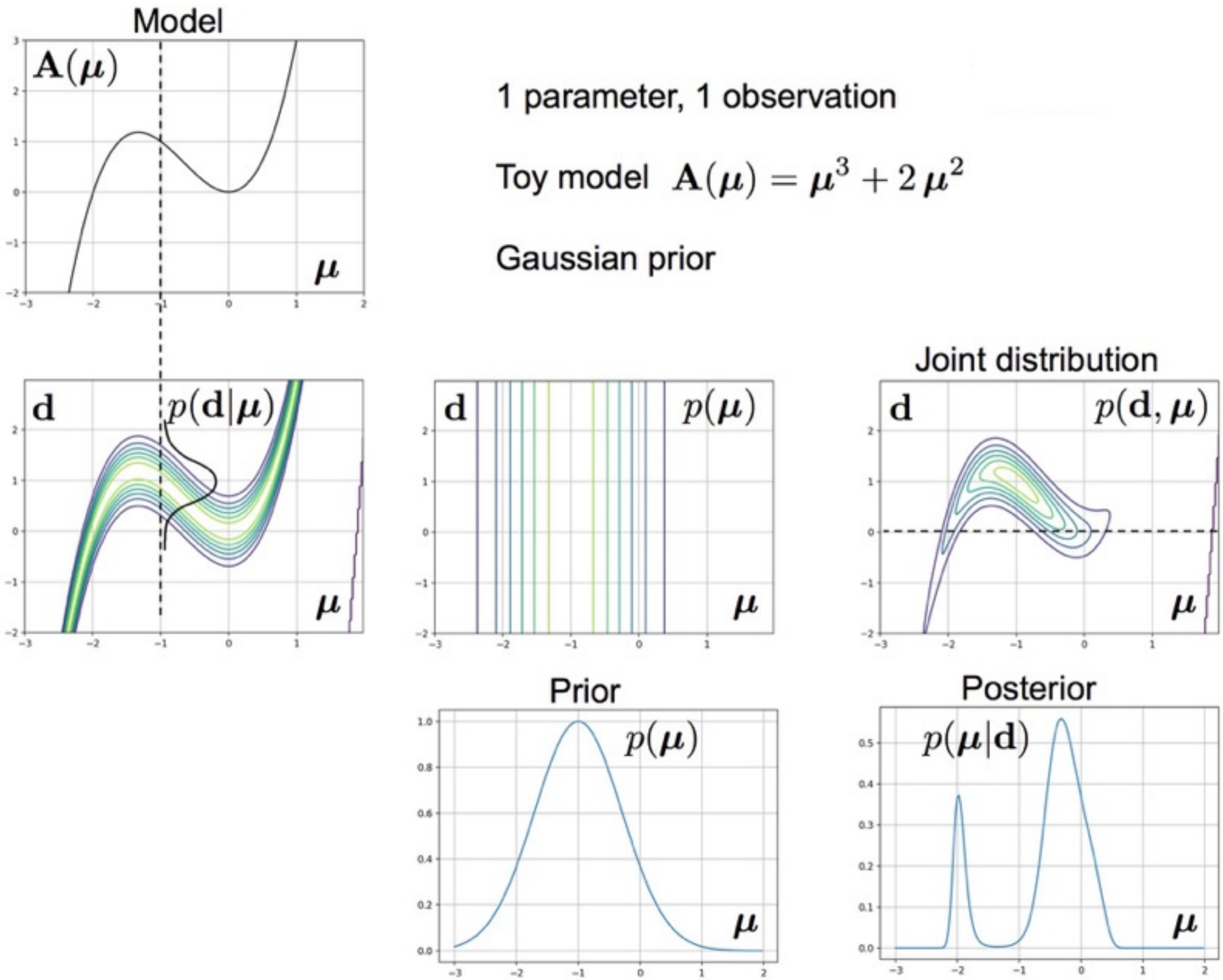
$$p(\boldsymbol{\mu}|\mathbf{d}) \propto \exp\left(-\frac{1}{2}(\mathbf{d} - \mathbf{A}(\boldsymbol{\mu}))^T\boldsymbol{\Sigma}^{-1}(\mathbf{d} - \mathbf{A}(\boldsymbol{\mu}))\right)p(\boldsymbol{\mu})$$

Bayesian Inverse Problem

1 parameter, 1 observation

Toy model $\mathbf{A}(\boldsymbol{\mu}) = \boldsymbol{\mu}^3 + 2\,\boldsymbol{\mu}^2$

Gaussian prior



Link with Deterministic Regularization

$$p(\boldsymbol{\mu}|\mathbf{d}) \propto p(\mathbf{d}|\boldsymbol{\mu})p(\boldsymbol{\mu})$$

$$p(\mathbf{d}|\boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^m|\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mathbf{A}(\boldsymbol{\mu}))^T\boldsymbol{\Sigma}^{-1}(\mathbf{d}-\mathbf{A}(\boldsymbol{\mu}))\right) \propto \exp\left(-\tilde{E}(\boldsymbol{\mu})\right)$$

$$p(\boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^m|\boldsymbol{\Sigma}_0|}} \exp\left(-\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\right) \propto \exp\left(-E_0(\boldsymbol{\mu})\right)$$

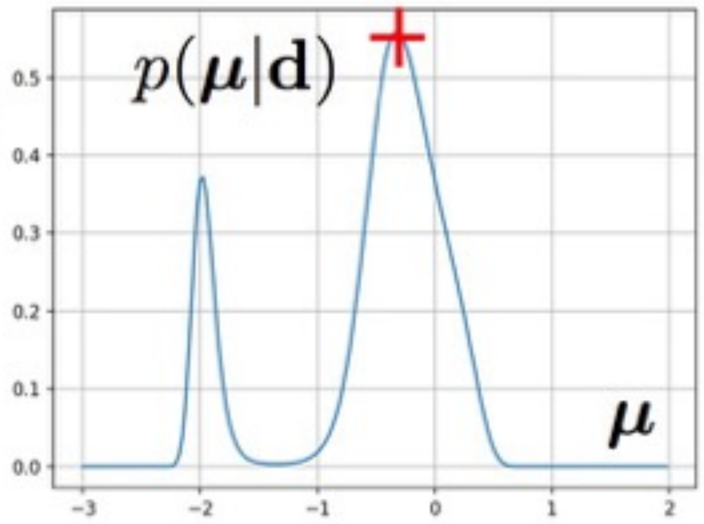
$$-\log p(\boldsymbol{\mu}|\mathbf{d}) = \tilde{E}(\boldsymbol{\mu}) + E_0(\boldsymbol{\mu}) + \mathrm{cte}$$

The log of the posterior distribution is the Tikkonov-regularized objective function of the deterministic inverse problem when using diagonal covariances

$$\boldsymbol{\mu}^\star = \arg\min_{\boldsymbol{\mu}}\left(-\log p(\boldsymbol{\mu}|\mathbf{d})\right) = \arg\min_{\boldsymbol{\mu}}\left(\tilde{E}(\boldsymbol{\mu}) + E_0(\boldsymbol{\mu})\right)$$

is called the Maximum A Posteriori estimate (MAP)

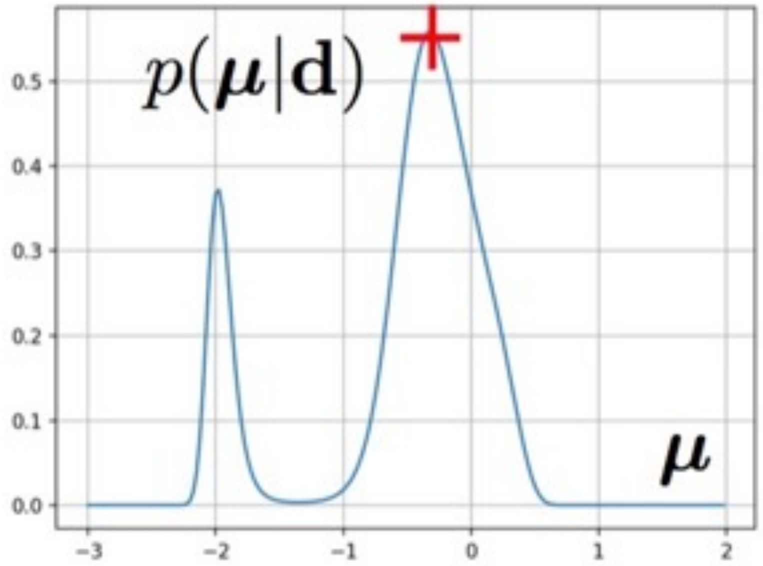
Gaussian priors are not restrictive as other priors may be mapped to Gaussian PDFs by changes of variable



Why all the trouble then?

- We have a distribution, with several potential modes

  → Variance tells us how much we have learned about model parameters



- We can propagate uncertainties to non-observed parts of the model, perform robust optimisation and/or control

- We can select models (including noise and prior parameters) in a principled, data-driven approach. This is done by maximising evidence

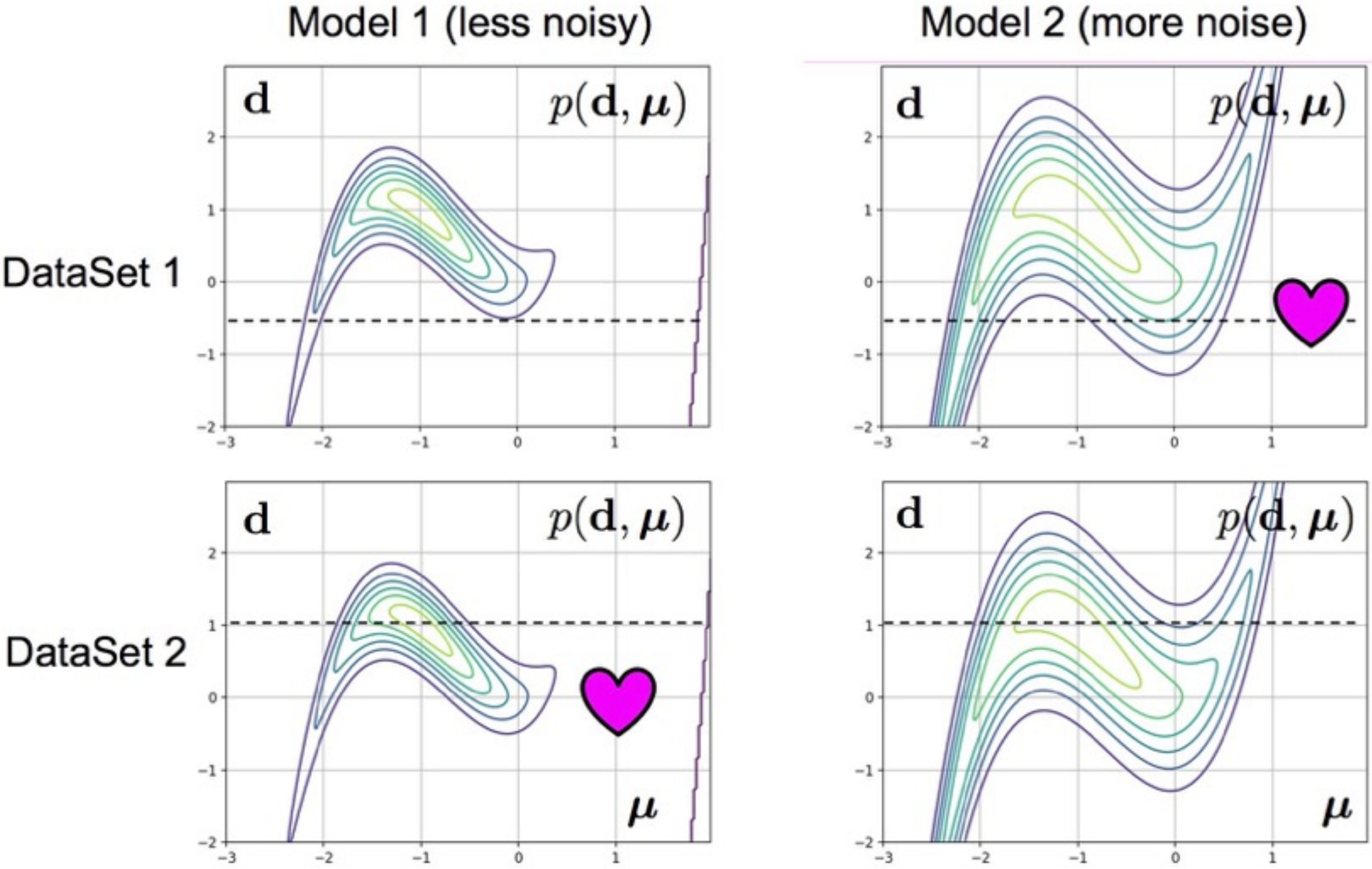
$$p(\mathbf{d}) = \int p(\boldsymbol{\mu}, \mathbf{d})\, d\boldsymbol{\mu}$$

- We can develop powerful design of experiments techniques to greedily minimise uncertainty

Bayesian Model Selection

- Maximise model evidence $p(\mathbf{d}) = \int p(\boldsymbol{\mu}, \mathbf{d})\, d\boldsymbol{\mu}$

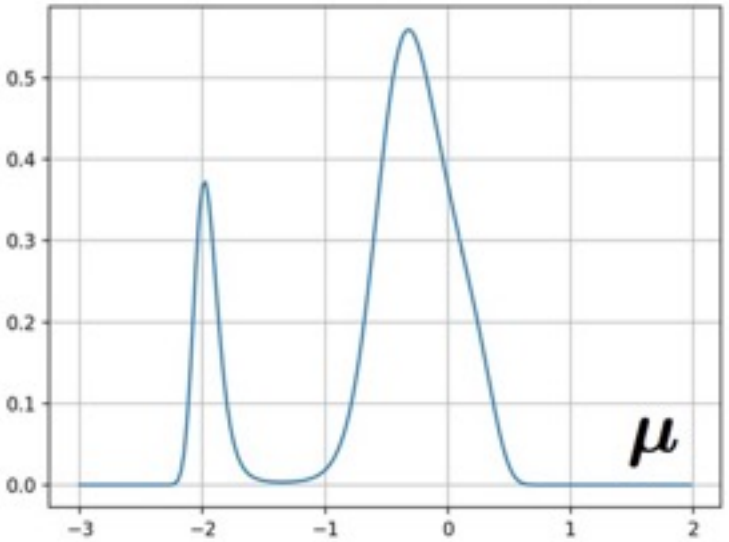
  → "which model is the most likely to generate the dataset?"



Sampling Posterior PDF

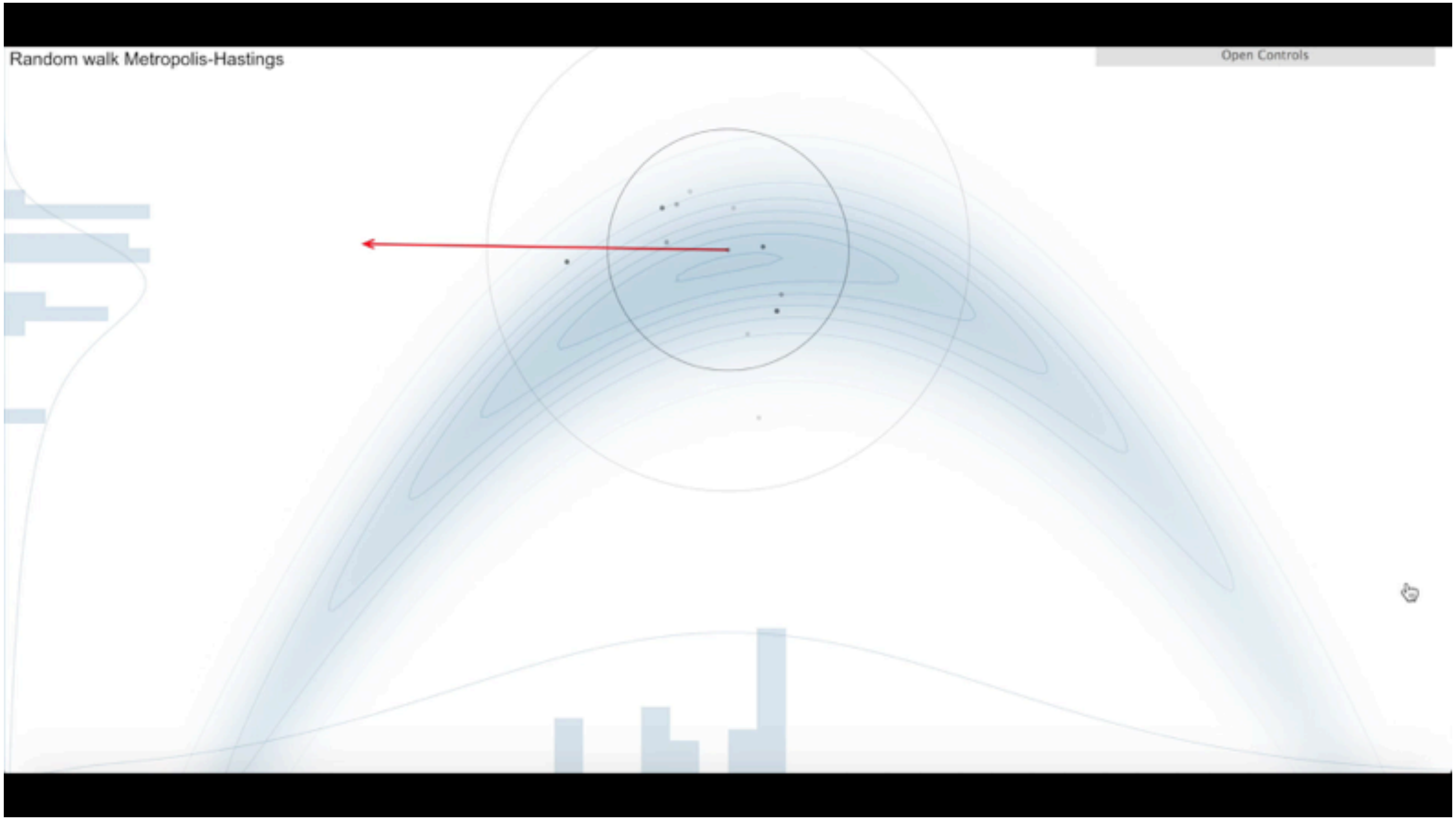
- Why not do this all the time then?
  - Answer 1: Evaluating the posterior at a point requires computing the FE model. Exploring exhaustively the distribution (*i.e.* more than for the MAP), is costly and algorithmically complicated.
  - Answer 2: Marginalisation of the likelihood function is extremely hard to do accurately, and even more costly than sampling the posterior

$$p(\boldsymbol{\mu}|\mathbf{d}) \propto p(\mathbf{d}|\boldsymbol{\mu})p(\boldsymbol{\mu})$$



- Worth it if uncertainty quantification is of prime interest to you, if you are to set up a method that will be reused in the future, or if you have strong priors and polluted and/or insufficient observations (e.g. model updating, sequential data assimilation)

MCMC



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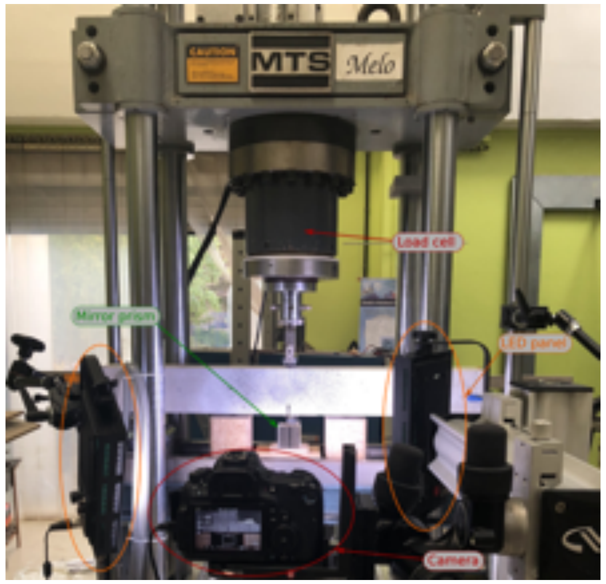
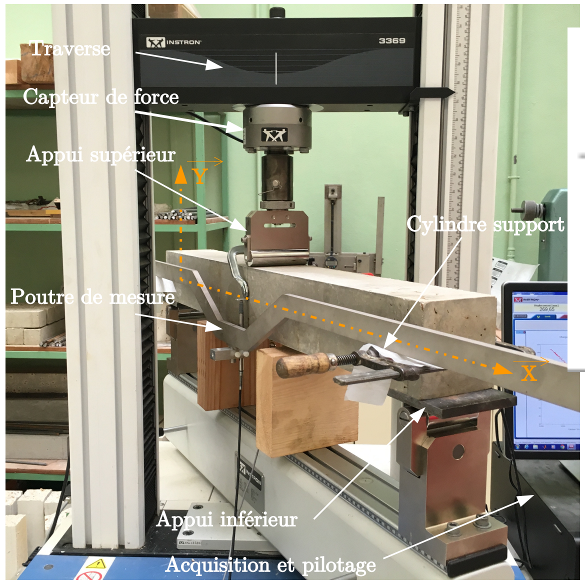
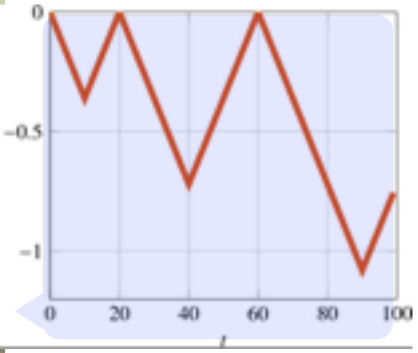
Real-case Illustration

Structural integrity on a large-scale damageable concrete structure

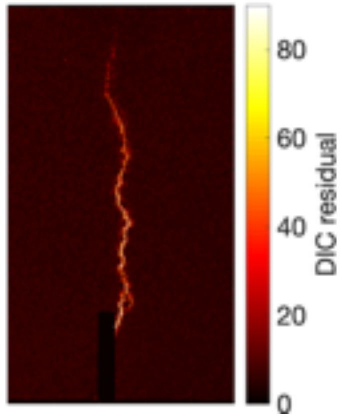


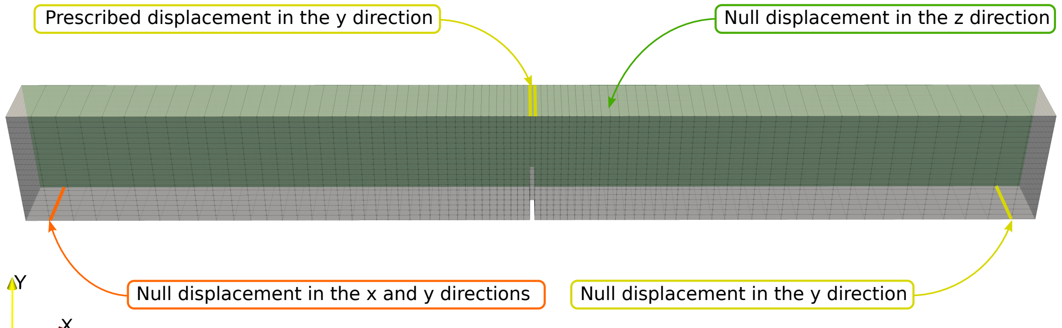


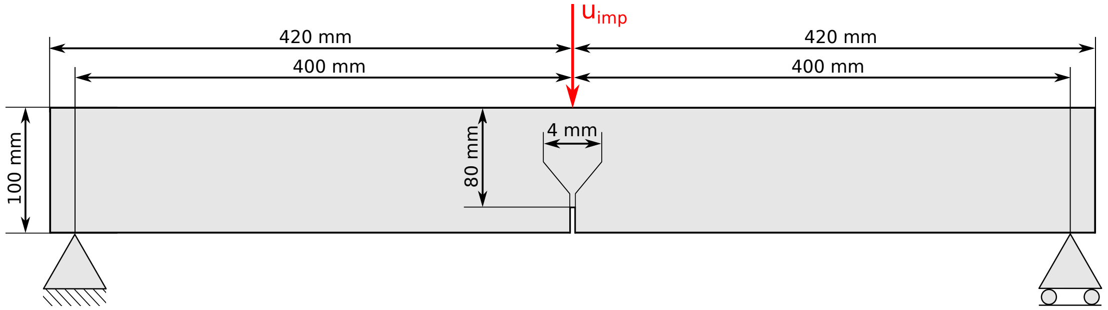


Ditical Image Correlation (DIC)

- → DIC pictures taken every 5s, and post-processed with Corelli [Leclerc *et al.* 15]
- → prediction of crack propagation & failure (before the physics!!)
- → simulator using an isotropic damage model







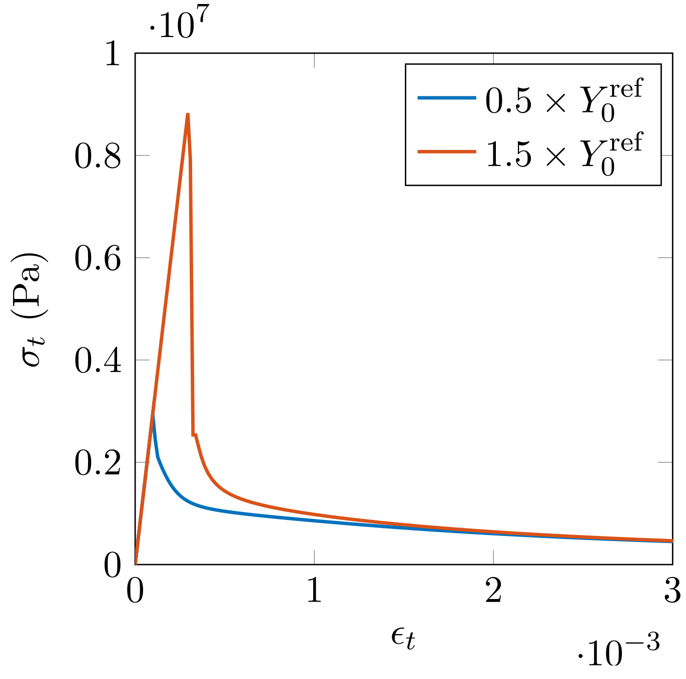
Objectives

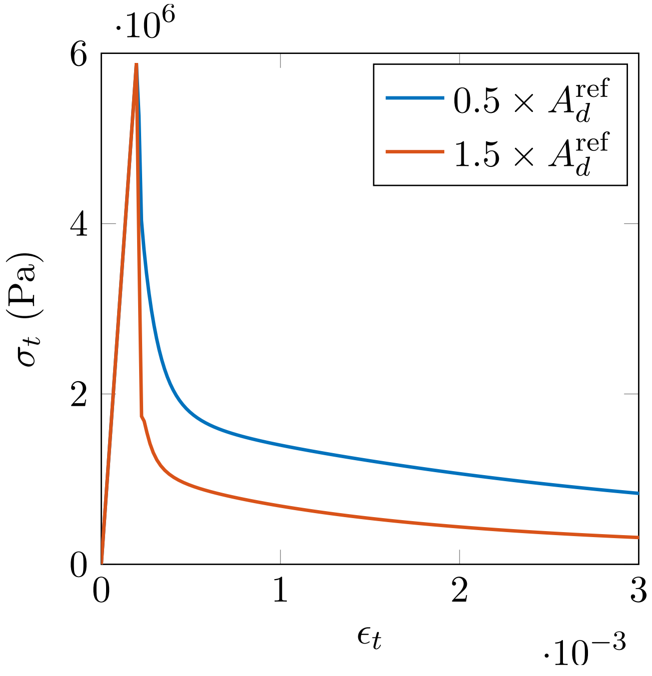
$$\boxed{\sigma = (1-d)\mathbf{C}\epsilon \quad ; \quad d(Y, A_d, Y_0) = 1 - \frac{1}{1 + A_d(Y - Y_0)}}$$

$Y = \frac{1}{2}\langle\epsilon\rangle_+ : \mathbf{C} : \langle\epsilon\rangle_+$ released energy

Y_0 initial threshold for damage initiation

A_d scalar brittleness (post-peak behavior)





⟶ **updating of parameters** (Y_0, A_d) **from data**

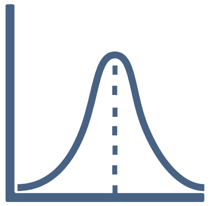
- robustness with uncertainties (measurement noise,…) ⟶ **Bayesian inference** [Kaipio & Sommersalo 04]
- real-time constraint ⟶ **Reduced Order Modeling (PGD)** [Chinesta *et al.* 14]
  ⟶ **Transport Map sampling** [El Moseley & Marzouk 12]
- model bias (BC, material (drying effects),…) ⟶ **online data-based correction**

Bayesian Formulation

[Kaipio & Sommersalo 04, Tarantola 05, Stuart 10]

- **Problem unknown**

$$\boxed{\pi(\mathbf{p}|\mathbf{d}^{\mathrm{obs}})}$$

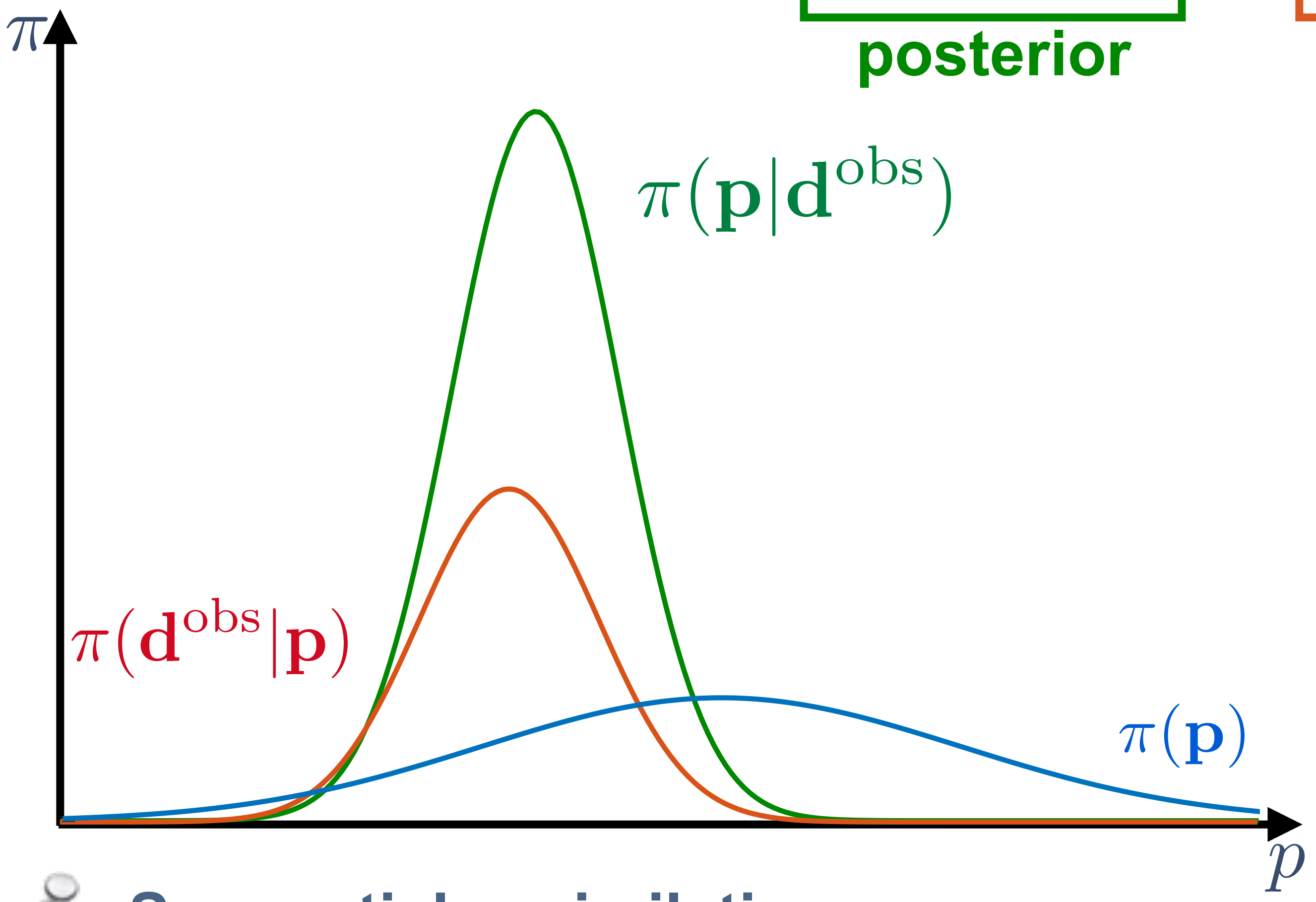


→ automatic regularization

→ natural framework to consider uncertainties

- **Bayes Theorem:**

$$\underset{\text{posterior}}{\boxed{\pi(\mathbf{p}|\mathbf{d}^{\mathrm{obs}})}} \propto \underset{\text{likelihood}}{\boxed{\pi(\mathbf{d}^{\mathrm{obs}}|\mathbf{p})}}.\underset{\text{prior}}{\boxed{\pi(\mathbf{p})}}$$



$$= \pi_{\mathrm{meas}}(\mathbf{d}^{\mathrm{obs}} - \mathbf{d}(\mathbf{p}))$$

if additive measurement noise

costly multi-query process

- **Sequential assimilation**

$$\pi(\mathbf{p}|\mathbf{d}_1^{\mathrm{obs}}, ..., \mathbf{d}_i^{\mathrm{obs}}) \propto \left(\prod_{j=1}^{i} \pi_{t_j}(\mathbf{d}_j^{\mathrm{obs}}|\mathbf{p})\right).\pi_0(\mathbf{p}) \qquad \pi_{t_j}(\mathbf{d}_j^{\mathrm{obs}}|\mathbf{p}) = \pi_{\mathrm{meas}}\left(\mathbf{d}_j^{\mathrm{obs}} - \mathbf{d}\left(\mathbf{p}, t_j\right)\right)$$

PGD Model Reduction

- modal description of multiparametric solution [Nouy 10, Chinesta *et al.* 14]
- low-rank canonical tensor format (separated variables)

$$\boxed{\mathbf{u}_m(\mathbf{x}, t, \mathbf{p}) = \sum_{k=1}^{m} \mathbf{\Lambda}_k(\mathbf{x}) \lambda_k(t) \prod_{i=1}^{d} \alpha_k^i(p_i)}$$

⟶ explicit dependency on parameters (extra-coordinates)

⟶ linear growth of ndofs with the number of parameters (computation/storage)

⟶ construction in the *offline* phase

⟶ use in the *online* phase of inference (multi-query computations)

⟶ straightforward model evaluation in the likelihood function

$\mathbf{d}(\mathbf{p}, t) = \mathcal{O}\left(\mathbf{u}_m(\mathbf{x}, t, \mathbf{p})\right)$ [Berger *et al.* 17, Rubio *et al.* 18]

⟶ may provide an explicit formulation of the posterior density

⟶ fast UQ on outputs of interest $q(\mathbf{p}) \approx \mathcal{Q}\left(\mathbf{u}_m(\mathbf{x}, t, \mathbf{p})\right)$

⟶ samples $q_k = q(\mathbf{p}_k)$

PGD Modes

$$\mathbf{u}_m(\mathbf{x}, t, Y_0, A_d) = \sum_{k=1}^{m} \mathbf{\Lambda}_k(\mathbf{x})\lambda_k(t)\alpha_k^1(Y_0)\alpha_k^2(A_d)$$

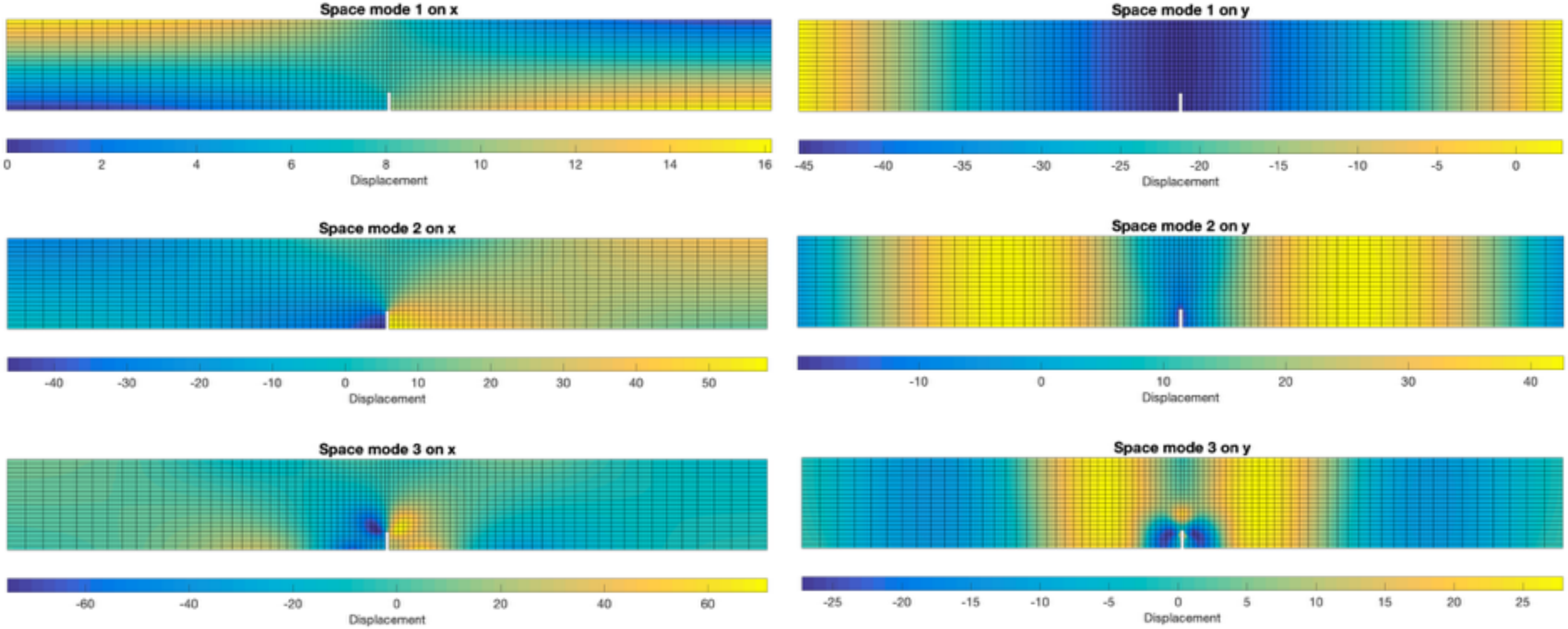
100 loading steps

$E = 30\ GPa \qquad \nu = 0.23$

$Y_0^{\mathrm{ref}} = 216\ J.m^{-3}$

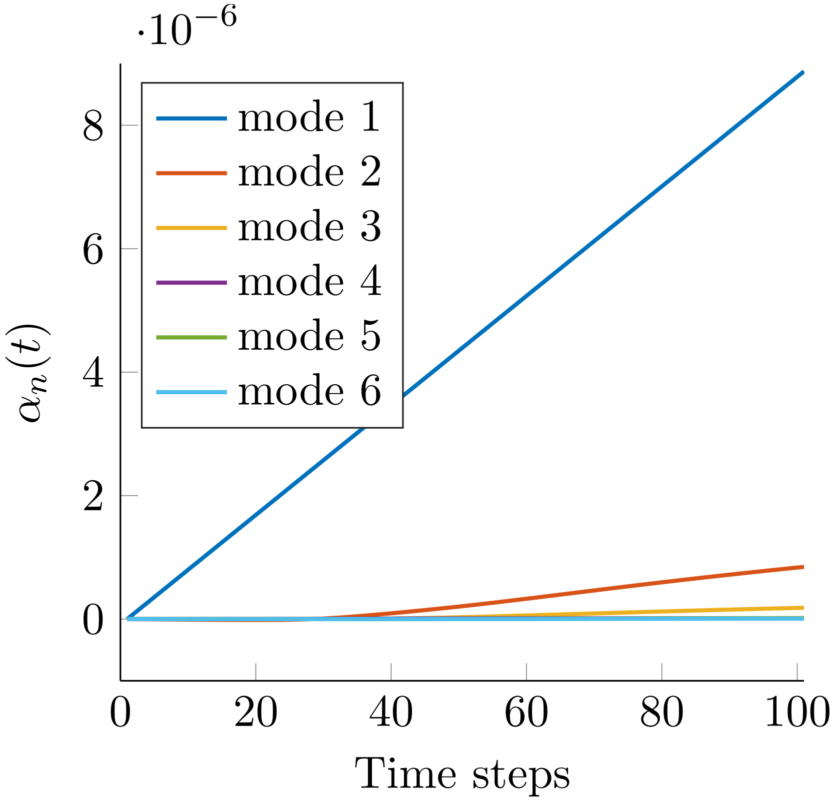
$A_d^{\mathrm{ref}} = 2.25\ J^{-1}.m^3$

- Space modes $\mathbf{\Lambda}_k$

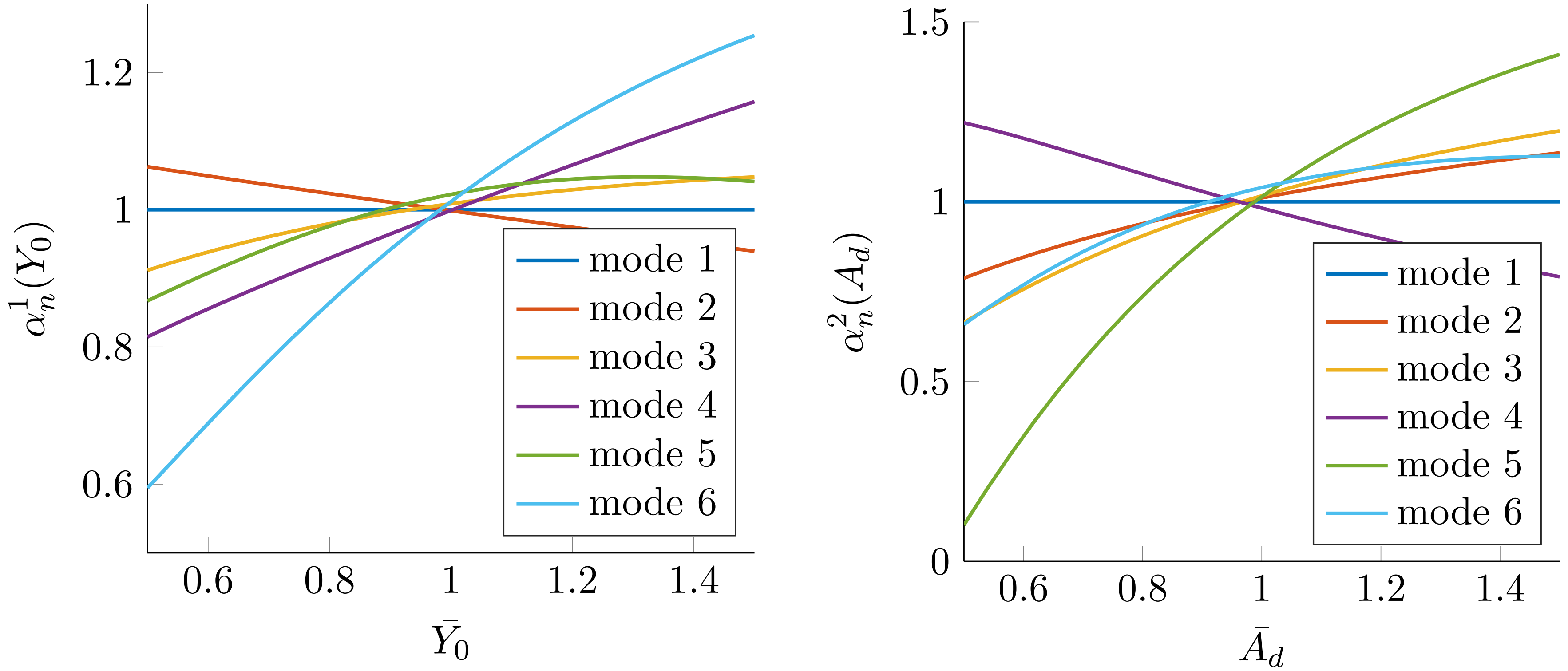


PGD Modes

- Time modes λ_k



- Parameter modes (α_k^1, α_k^2) (with $\bar{Y}_0 = Y_0/Y_0^{\text{ref}}, \bar{A}_d = A_d/A_d^{\text{ref}}$)



→

1st mode

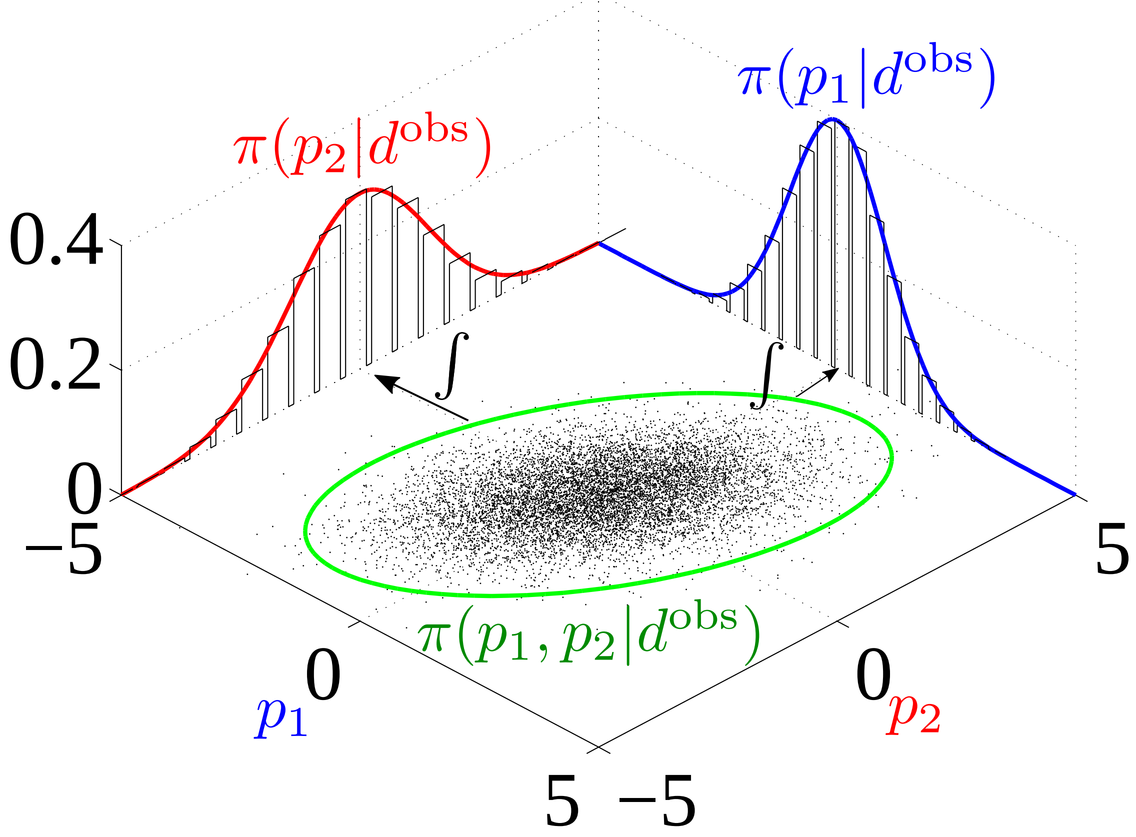
=

full elasticity mode

Sampling Posterior PDFs

Characterization/exploration of the posterior density

- Mean *a posteriori*
- Maximum *a posteriori*
- 1D Marginals
- Uncertainty propagation



➡ Need multi-dimensional integration

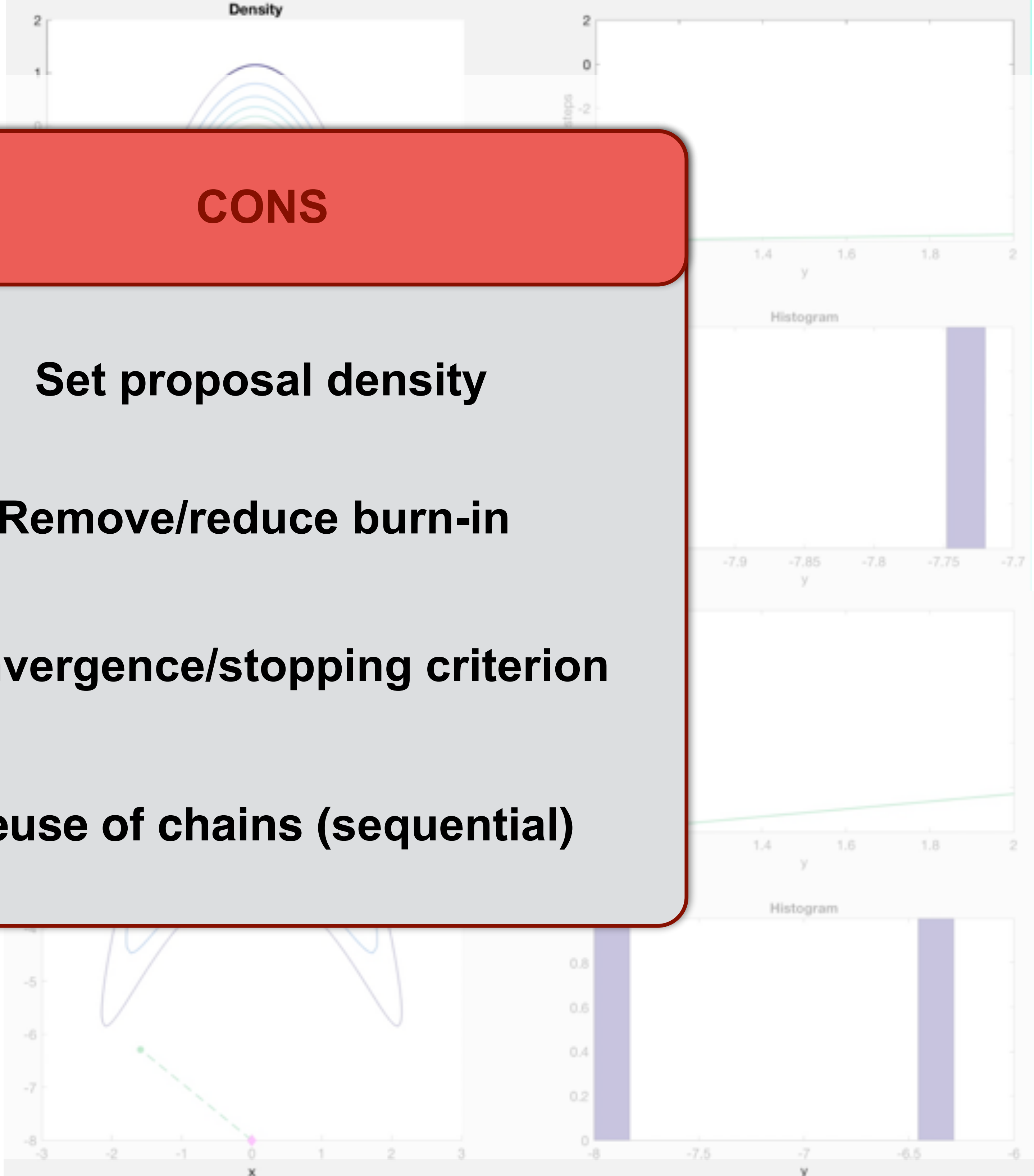
Monte-Carlo integration:

- Quantity of interest: $\mathbb{E}[h] = \int h(\mathbf{p})\pi(\mathbf{p})d\mathbf{p}$
- With samples $\mathbf{p}^{\{1,\dots,N\}} \sim \pi \quad \mathbb{E}[h] \approx \overline{h} = \frac{1}{N}\sum_{i=1}^{N} h(\mathbf{p}^i)$

➡ Need samples from posterior density

Sampling Posterior PDFs

- **Markov Chain Monte-Carlo (MCMC) method**



- **small** proposal

(chain stays confined)

CONS

Set proposal density

Remove/reduce burn-in

Convergence/stopping criterion

Reuse of chains (sequential)

- **large** proposal

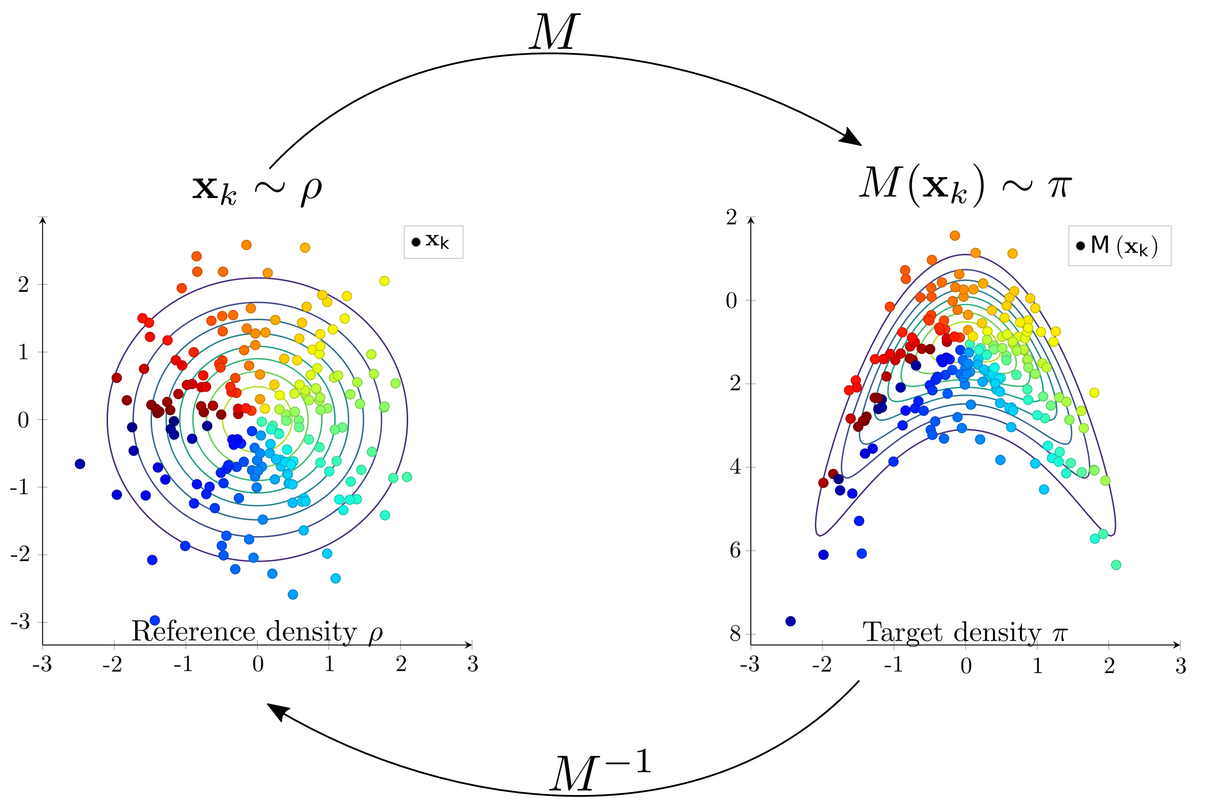
(increased rejection rate)

Transport Map Sampling

- **Transport Maps method** [Villani 07 (optimal transport), El Moselhy & Marzouk 12]

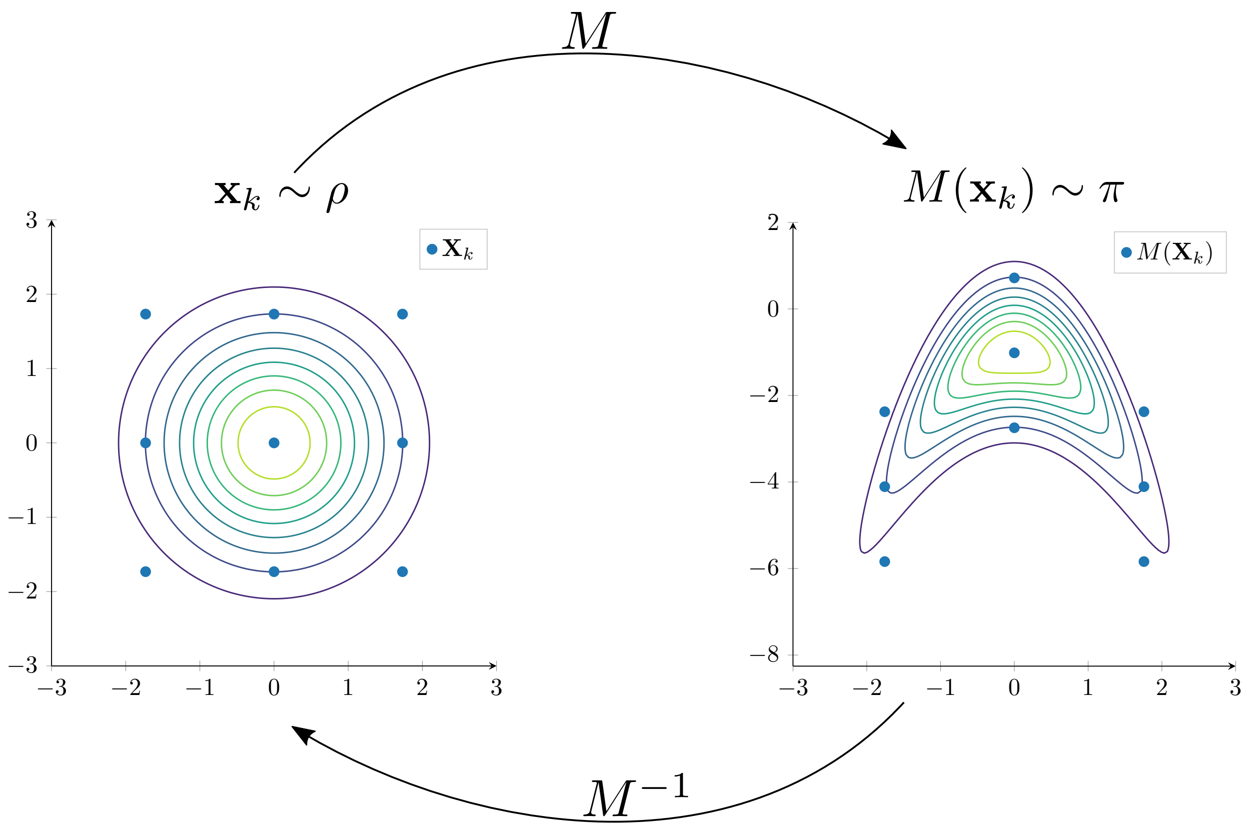
Transport integrals over the **target** density to integrals over a **reference** density

$$\boxed{\int g \mathrm{d}\nu_\pi = \int g \circ \underset{\substack{\downarrow \\ \text{deterministic application}}}{M} \mathrm{d}\nu_\rho}$$



transport of samples

(pushing forward Gaussian samples through the map)



transport of quadrature rules

→ computations (sampling, integration) performed in the reference space [Marzouk 16, Cui & Dolgov 20]

Transport Map Sampling

Parametrization of Transport Maps

- **Structure of the class:**

$$M(\mathbf{p}) = \begin{bmatrix} M^1(\mathbf{a}_c^1, \mathbf{a}_e^1, p_1) \\ M^2(\mathbf{a}_c^2, \mathbf{a}_e^2, p_1, p_2) \\ \vdots \\ M^d(\mathbf{a}_c^d, \mathbf{a}_e^d, p_1, p_2, ..., p_d) \end{bmatrix}$$

Knothe-Rosenblatt rearrangements
(lower triangular monotonic maps)

- unique minimizer
- computational tractable, invertible
- optimality for a weighted metric

[El Moselhy & Marzouk 12]
[Papamakarios *et al.* 19]

- **Parametrization:**

$$M^k(\mathbf{a}_c^k, \mathbf{a}_e^k, \mathbf{p}) = \Phi_c(\mathbf{p})\mathbf{a}_c^k + \int_0^{p_k} (\Phi_e(p_1, ..., p_{k-1}, \theta)\mathbf{a}_e^k)^2 \mathrm{d}\theta$$

Φ_c, Φ_e : Hermite polynomials which given order

$\mathbf{a}_c, \ \mathbf{a}_e$: parameters

→ obtained from minimization of Kullback-Liebler divergence

$$\boxed{\mathcal{D}_{KL}(M_\sharp \nu_\rho || \nu_\pi) = \mathbb{E}_\rho \left[\log \frac{\nu_\rho}{M_\sharp^{-1} \nu_\pi} \right]}$$

push forward operator

requires unnormalized pdfs alone

Transport Map Sampling

Minimization problem

$$\min_{\mathbf{a}_c^{1,\dots,d},\mathbf{a}_e^{1,\dots,d}} \sum_{i=1}^{N} \omega_i \left[-\log(\tilde{\pi}\circ M(\mathbf{a}_c^{1,\dots,d},\mathbf{a}_e^{1,\dots,d},\mathbf{p}_i) - \log(|\det\nabla M(\mathbf{a}_c^{1,\dots,d},\mathbf{a}_e^{1,\dots,d},\mathbf{p}_i))|)\right]$$

$\tilde{\pi}(\mathbf{p}|\mathbf{d}^{\mathrm{obs}}) = \pi_{\mathrm{meas}}(\mathbf{d}^{\mathrm{obs}} - u_m(\mathbf{p}))\pi(\mathbf{p})$ (non-normalized pdf)

→ solved with gradient/Hessian information (BFGS,…)

→ partial derivatives explicitly recovered and stored in the **offline phase**

$$\frac{\partial^n \mathbf{u}_m}{\partial p_j^n}(\mathbf{x},t,\mathbf{p}) = \sum_{k=1}^{m} \mathbf{\Lambda}_k(\mathbf{x})\lambda_k(t)\frac{\partial^n \alpha_k^j}{\partial p_j^n}(p_j)\prod_{\substack{i=1\\ i\neq j}}^{d}\alpha_k^i(p_i)$$

→ large speed-up for the computation of maps!!! [Rubio *et al.* 19]

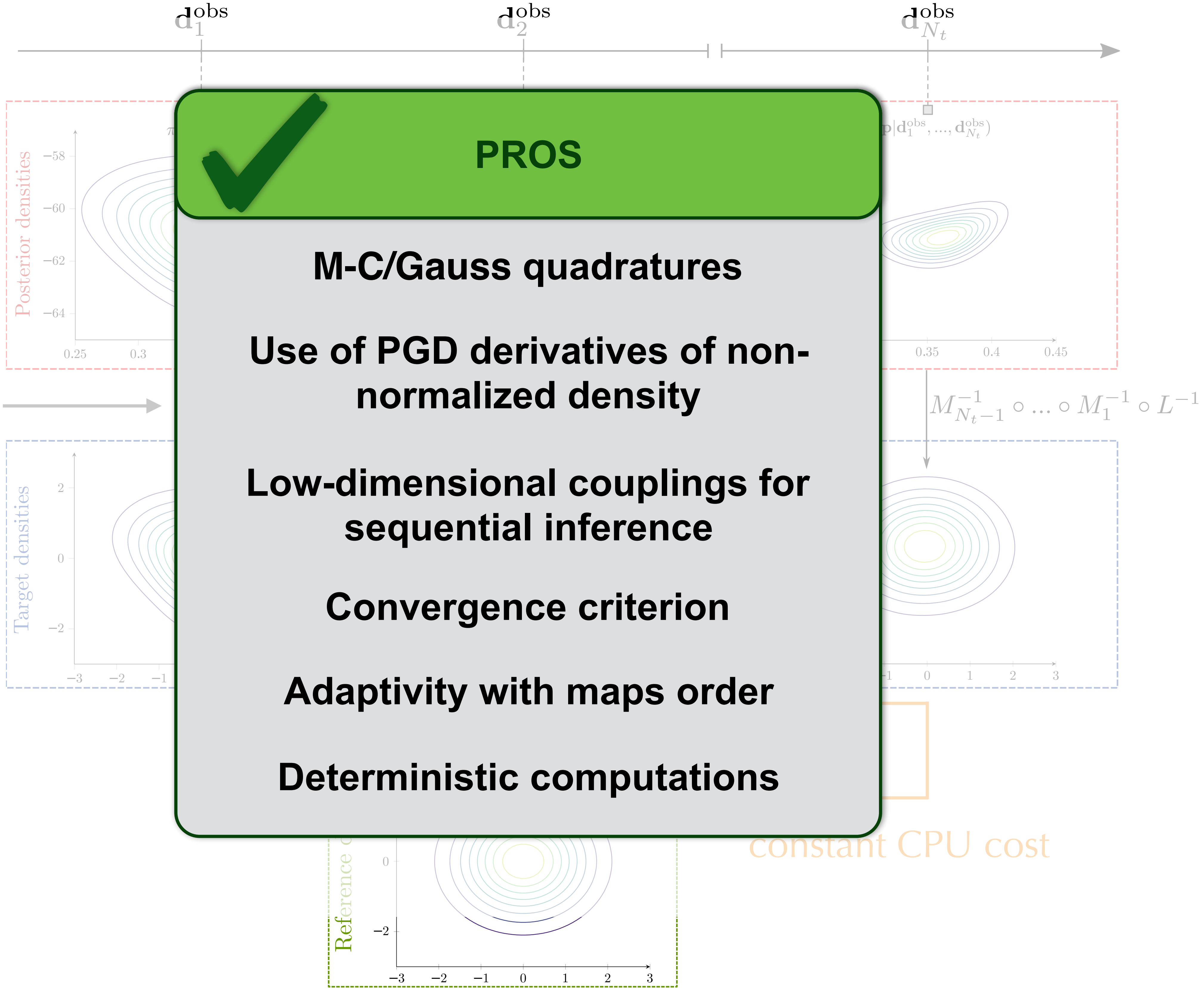
Variance diagnostic [Spantini *et al.* 18]

$$\epsilon_\sigma = \frac{1}{2}\mathbb{V}\mathrm{ar}_\rho\left[\ln\frac{\nu_\rho}{M_\sharp^{-1}\nu_\pi}\right]$$

- sampling error estimate
- clear convergence criterion
- adaptive strategy on map order

Transport Map Sampling

- Sequential updating (composition of maps)



Laplace approximation *(linear map for normalization & local Gaussian approximation from gradient and Hessian)*

PROS

M-C/Gauss quadratures

Use of PGD derivatives of non-normalized density

Low-dimensional couplings for sequential inference

Convergence criterion

Adaptivity with maps order

Deterministic computations

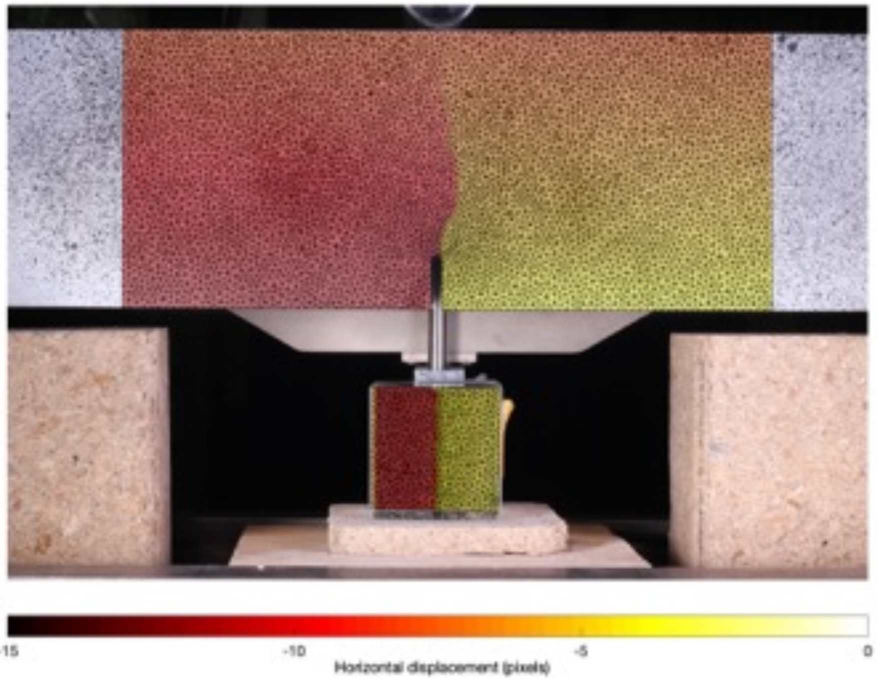
Results

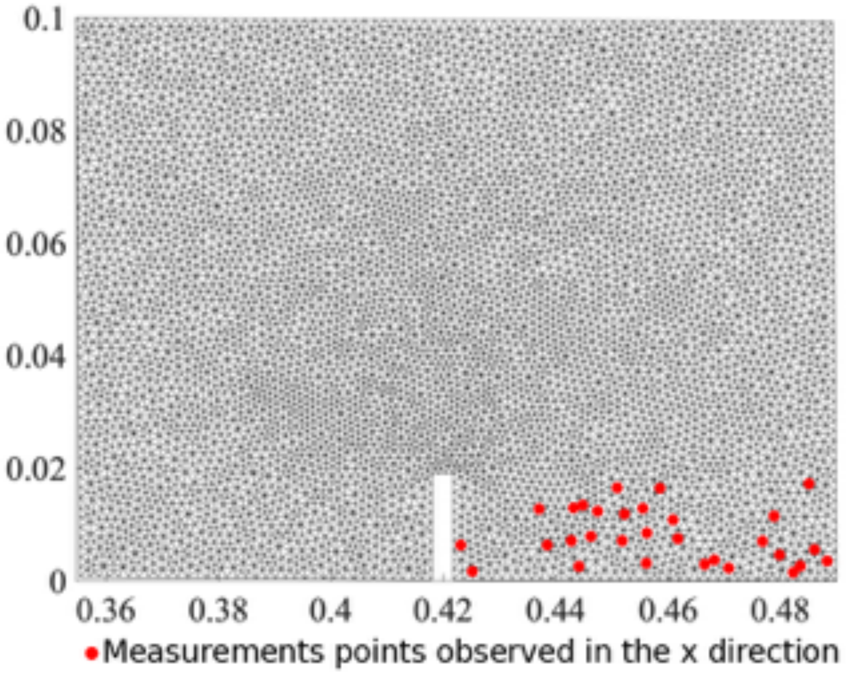
Assimilation with Transport Maps & PGD

$$\pi(\bar{Y}_0, \bar{A}_d | \mathbf{d}_1^{\text{obs}}, \ldots, \mathbf{d}_i^{\text{obs}}) \propto \prod_{j=1}^{i} \pi(\mathbf{d}_j^{\text{obs}} | \bar{Y}_0, \bar{A}_d).\pi_0(\bar{Y}_0, \bar{A}_d)$$

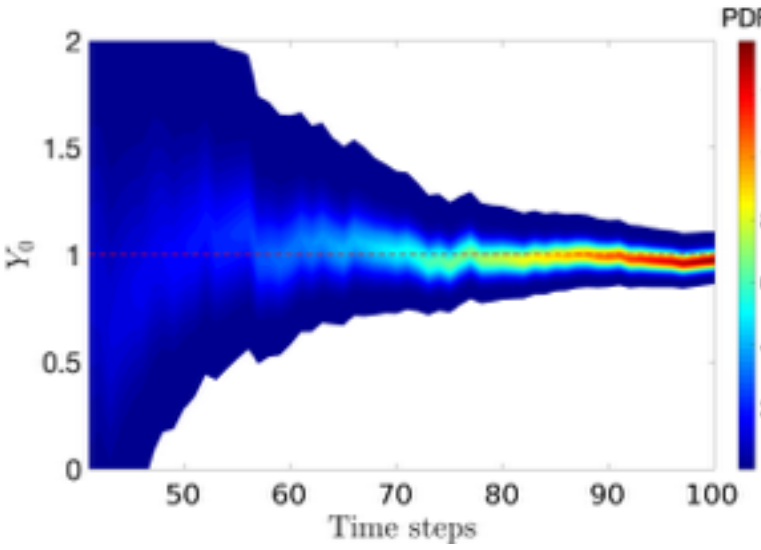
$$\epsilon_\sigma = 10^{-3}$$

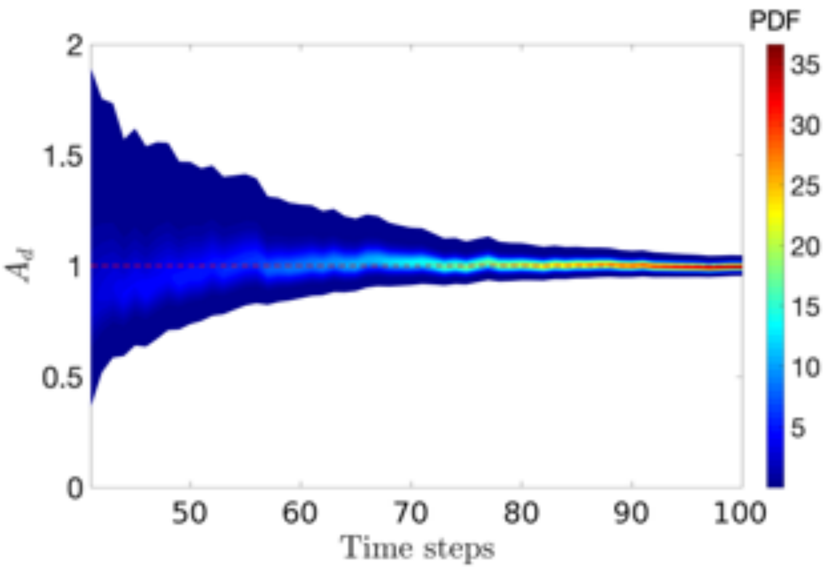
PGD approximation with $m = 6$

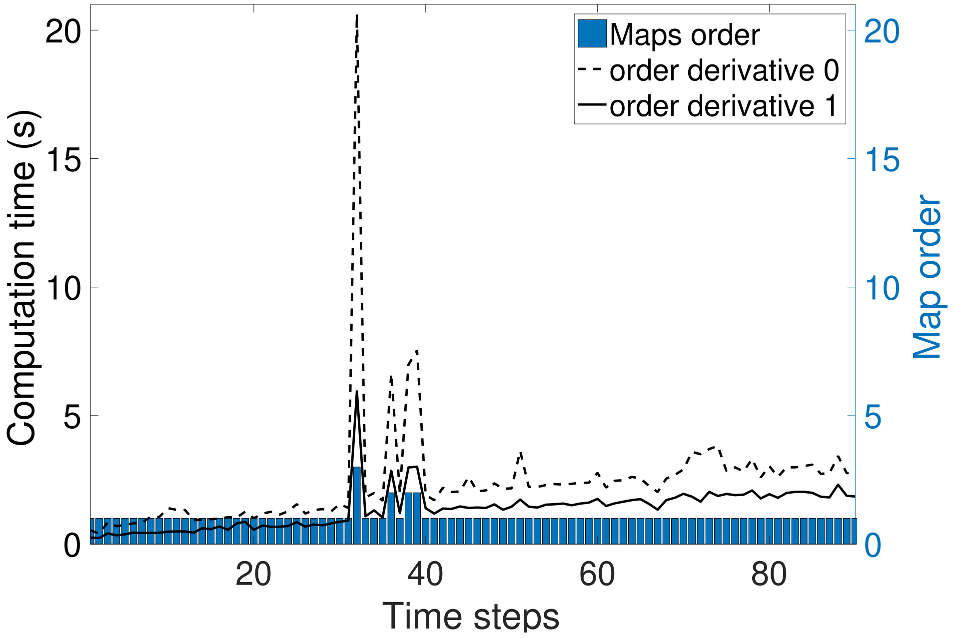




selection of most relevant DIC data (sensitivity analysis)



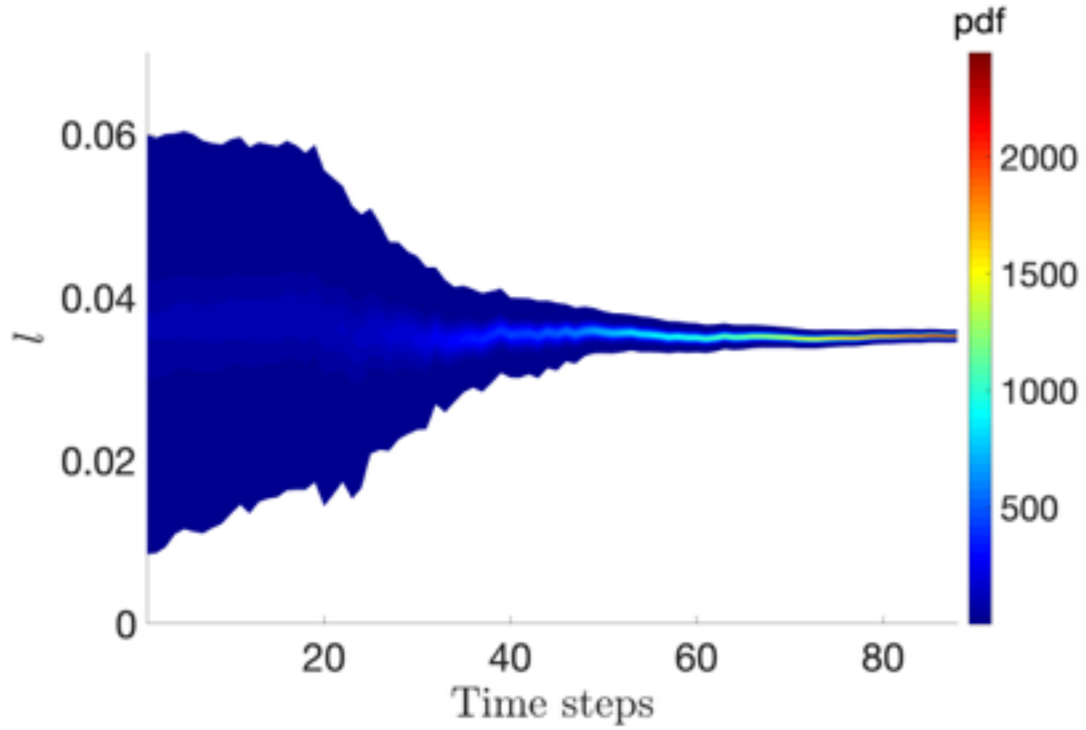


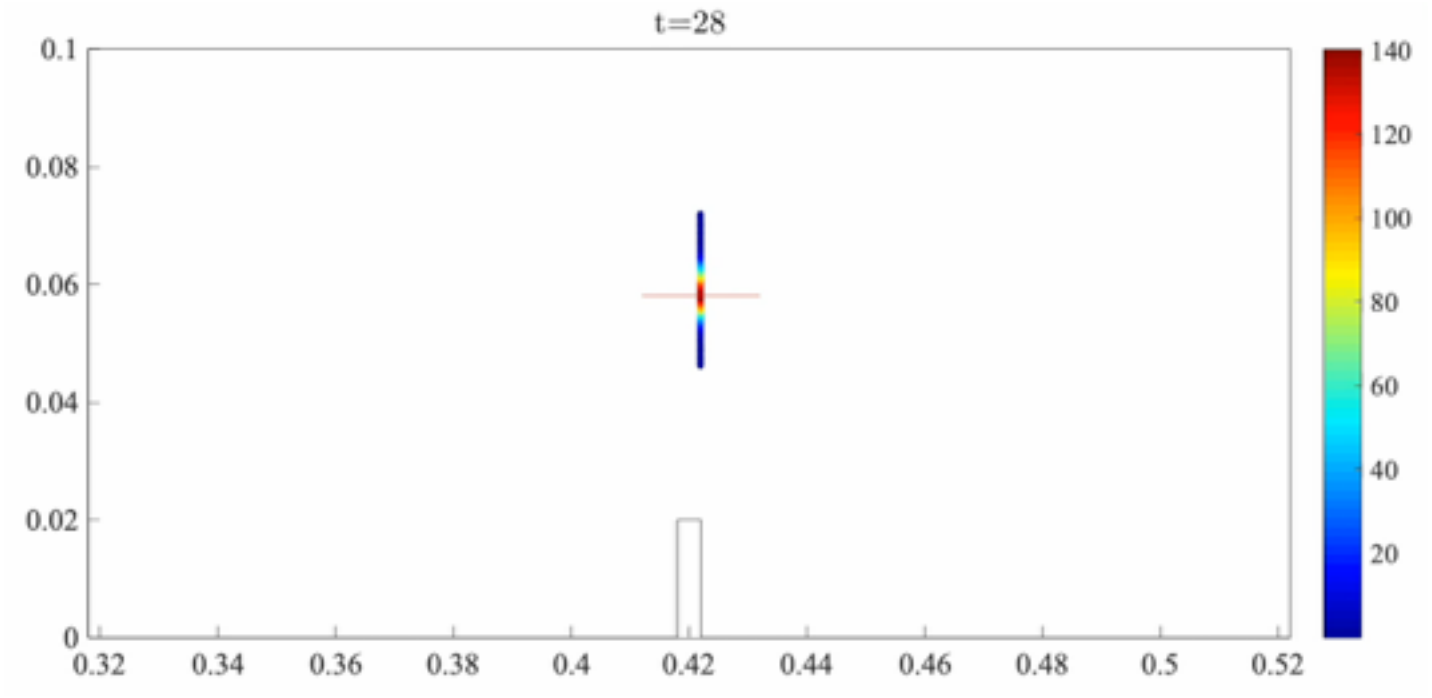


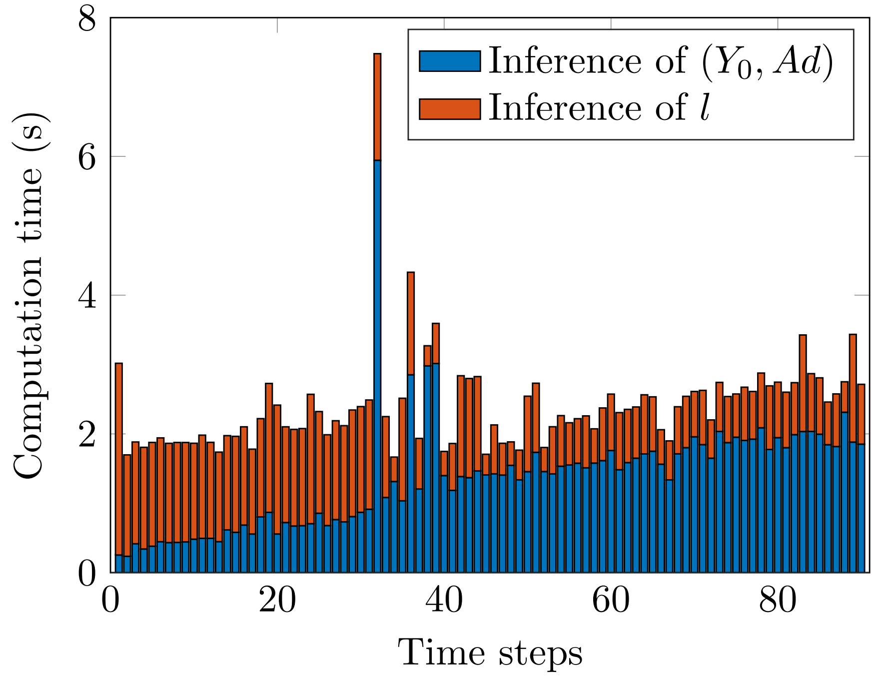
Post-processing

- **On-the-fly prediction of the final crack length l_T**

$$\boxed{\pi(l_T) = \pi_{\mathrm{u}}(\mathbf{u}^{\mathrm{SVD}}(l_T)).\pi_0(l_T)}$$







→ can be considered as real-time

→ possibility to further control the process with on-the-fy command synthesis

PB. Rubio, L.C., F. Louf, Real-time data assimilation and control on mechanical systems under uncertainties, *AMSES*, 8:4 (2021)

Model Bias Correction

→ **data-based enrichment**, comparing predicted outputs and actual data

→ defined dynamically and in a stochastic setting

extension of PBDW/hybrid twins [Maday *et al.* 15, Chinesta *et al.* 18]

- **Stochastic residual (computable)**

$$\mathbf{B}(\mathbf{x}^{\mathrm{obs}}, t_i) = \mathbf{d}_i^{\mathrm{obs}} - \mathbf{e}_{\mathrm{meas}} - \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, t_i, \mathbf{p})$$

spatial coordinates of measurement

- **Corrected model**

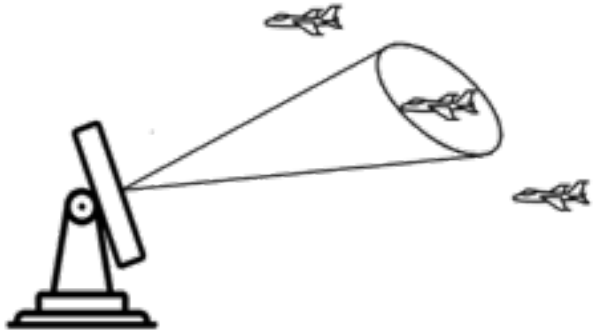
$$\mathcal{M}^{\mathrm{corr}}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}) = \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}) + \hat{\mathbf{B}}_{i\to i+1}(\mathbf{x}^{\mathrm{obs}})$$

extrapolated model bias

(Gaussian pdf + use of Sequential Karhunen-Loeve method [Ross et al. 08])

$$\boxed{\pi(\mathbf{d}_{i+1}^{\mathrm{obs}} | \mathbf{p}) = \pi_{\hat{B}}(\mathbf{d}_{i+1}^{\mathrm{obs}} - \mathcal{M}(\mathbf{x}^{\mathrm{obs}}, \mathbf{p}, t_{i+1}))}$$

Filtering



Consider a physical system with **state** x

- the state is usually not directly reachable
- it should be retrieved from (incomplete, noisy) **observations** y

GOAL: recover relevant information from observed signals ➡ role of an **observer**

CHALLENGE: assess at best the state of the system, from | observations
| assumptions on the system behavior

Filtering: extract the state at time t_k from all past and current observations $y_1, \ldots, y_k$

Prediction: forecast the futur state at time t_k from past observations $y_1, \ldots, y_{k-j}$

Smoothing: estimate the state at time t_k from past/current/future observations $y_1, \ldots, y_K$

Example: target tracking ➡ provide accurate continuously updated info on position/velocity

- given a sequence of partial noisy observations
- given the modeled dynamics governing time evolution

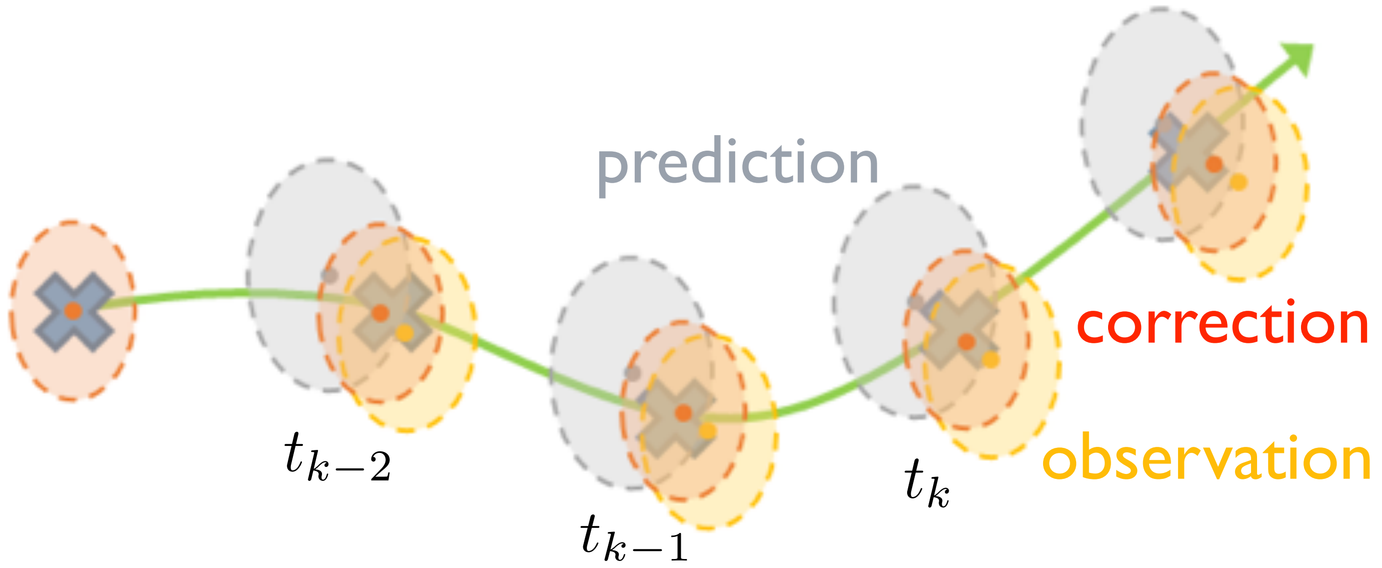
Many other applications: control, weather forecasting, economics,…

Kalman Filtering

[Kalman 60, Grewal 93]

- Set of mathematical equations that provides an efficient computational means to estimate x
  - minimizes the mean of the squared error (estimated error covariance) when some presumed conditions are met
  - enables estimations of past, present and future states, even when the precise nature of the modeled system is unknown
  - recursive process: output at each time t_k depends on observation at time t_k and output at time t_{k-1}

    → *no need to store the whole set of observations*
    constant CPU time (real-time implementation possible)
  - Two-step **prediction/correction** algorithm, using feedback from measurements



Kalman Filtering

[Kalman 60]

state

control input

Dynamical system (linear):

$$x_k = M_k x_{k-1} + B_k u_k + w_k$$
$$y_k = H_k x_k + v_k$$

observations

Gaussian white noise

Bayes theorem: $p(x_k|y_{1:k}) = p(x_k|y_k, y_{1:k-1}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})}$

under the assumptions:

- state x_k is a Markov process
- observations y_k are statistically independent of state history

recursive state estimate
(MAP for Gaussian PDFs)

prediction: $\hat{x}_{k|k-1} = M_k \hat{x}_{k-1|k-1} + B_k u_k$ (state estimate)

$P_{k|k-1} = M_k P_{k-1|k-1} M_k^T + Q_k$ (covariance matrix)

$\mathrm{E}[w_k w_k^T]$

innovation

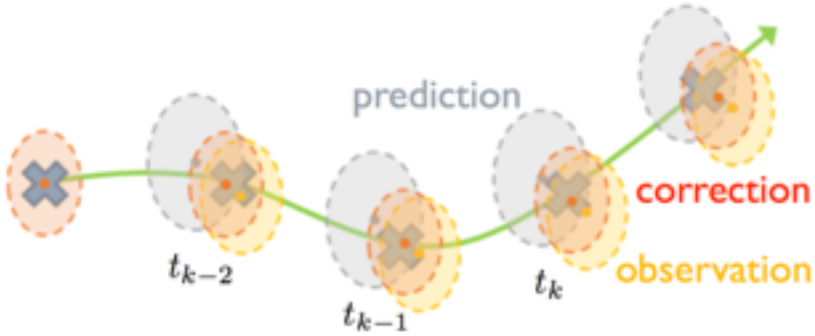
correction: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - H_k \hat{x}_{k|k-1})$

$P_{k|k} = (I - K_k H_k) P_{k|k-1}$

$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$

Kalman gain, weights innovation

$\mathrm{E}[v_k v_k^T]$



$$\lim_{\Sigma_v \to 0} K_k = H^{-1}$$

$$\lim_{P_k^- \to 0} K_k = 0$$

Simple Example

GOAL : Estimate the position/velocity of a plane moving in 1D with almost constant speed

$$x_k = x_{k-1} + \dot{x}_{k-1}\Delta t + \frac{1}{2}a_{k-1}\Delta t^2$$
$$\dot{x}_k = \dot{x}_{k-1} + a_{k-1}\Delta t$$
$$y_k = x_k + v_k \quad \text{with } v_k \sim \mathcal{N}(0, \sigma_v^2)$$

To model small variations of acceleration (that perturb the trajectory), we use Gaussian r.v.

We introduce the generalized state $X_k = \begin{pmatrix} x_k \\ \dot{x}_k \end{pmatrix}$

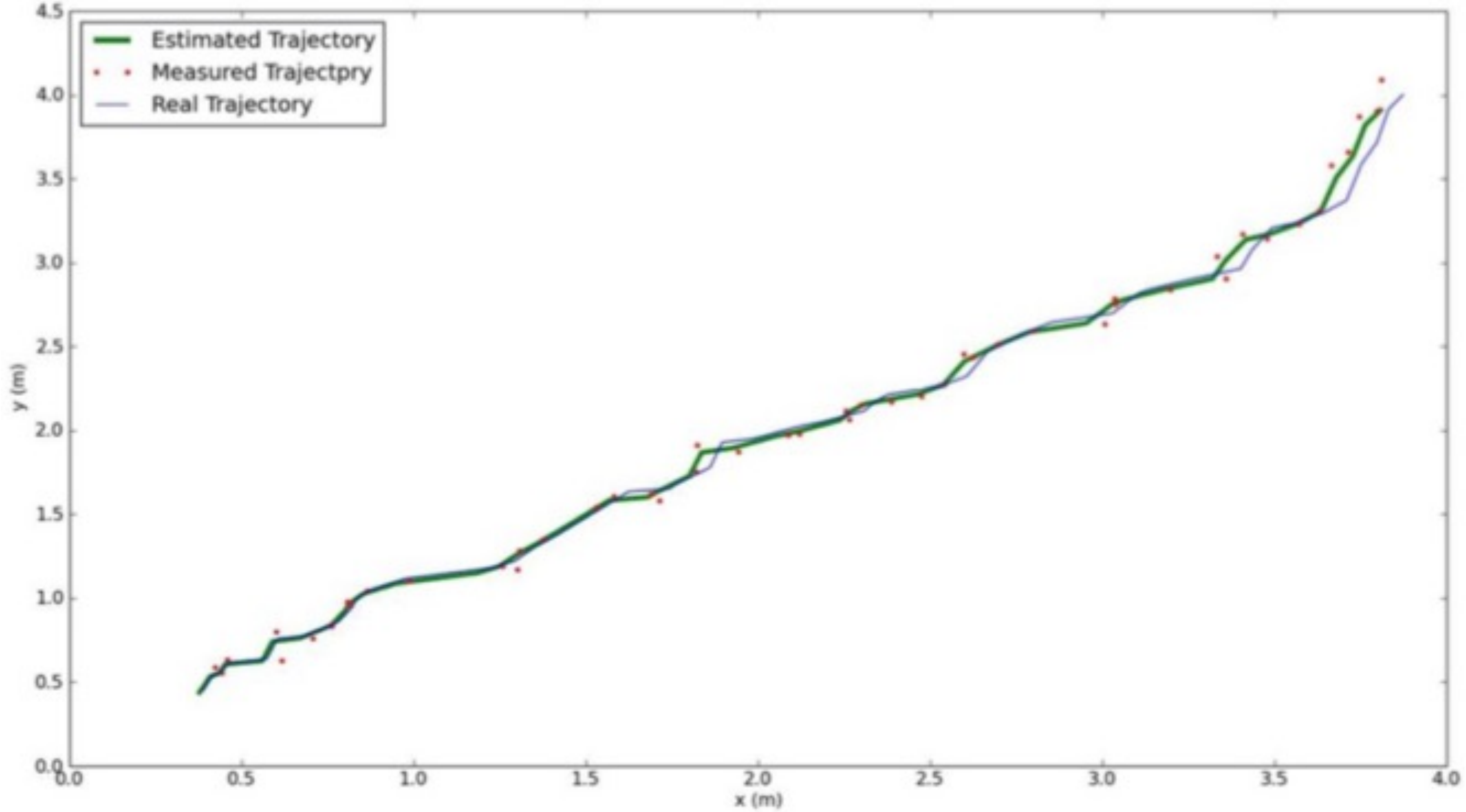
$$X_k = AX_{k-1} + w_k$$
$$y_y = HX_k + v_k$$

with $A = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$w_k = \begin{pmatrix} \Delta t^2/2 \\ \Delta t \end{pmatrix} a_{k-1} \sim \mathcal{N}(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_w) \text{ with } \Sigma_w = \begin{bmatrix} \Delta t^4/4 & \Delta t^3/2 \\ \Delta t^3/2 & \Delta t^2 \end{bmatrix} \sigma_a^2$$

Simple Example

Illustrative results (in 2D: $X_k = (x_{1k}\ x_{2k}\ \dot{x}_{1k}\ \dot{x}_{2k})^T$)



Kalman Filtering

Extension to nonlinear dynamical systems

$$x_k = M(x_{k-1}, u_k) + w_k$$
$$y_k = H(x_k) + v_k$$

➡

extended KF (EKF)
Particle Filter (or Sequential MC)
unscented KF (UKF) [Julier & Uhlmann 97]
…

UKF:
- deterministic selection of σ-points (Cholesky decomposition of cov. matrix)
- transformation of these points $\{x_i\}_{i=1,\ldots,2N+1}$ through nonlinear functions
  $\longrightarrow$ *parallel procedure* [Azam *et al.* 12]
- high accuracy (order 3), but strongly depends on data noise [Li *et al.* 16]

Extension to parameter estimation [Mariani & Corigliano 04, Moireau & Chapelle 11]

$$x_k = M(x_{k-1}, \theta, u_k) + w_k \longrightarrow$$ state + parameter estimation

Joint Kalman Filter (KF on extended state)	**Dual Kalman Filter** (combines 2 KFs)
$\theta_k = \theta_{k-1} + w_k^\theta$ (stationarity hyp.) $\overline{x}_k = [x_k^T, \theta_k^T]^T$ (concatenation)	$\theta_k = \theta_{k-1} + w_k^\theta$ (parameters as state vect.) $y_k = \underset{\uparrow}{H_{dual}}(x_k, \theta_k, u_k, w_k) + v_k$ observer = state evaluation operator (based on a second KF, for given θ_k)

$\longrightarrow$ nonlinear systems

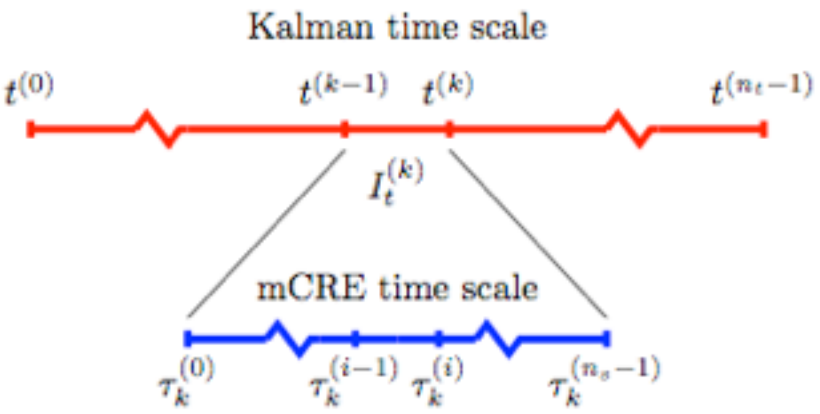
Modified Dual Kalman Filtering (MDKF)

[Marchand *et al.* 16, Diaz *et al.* 22]

IDEA: change the metric space of observer in Dual Kalman Filter

↳ UKF coupled with the mCRE functional (new observation operator in the update)

$$\begin{aligned} \theta_k &= \theta_{k-1} + w_k^\theta \\ 0 &= \nabla_\theta \mathcal{E}_{mCRE}(x_k, \theta_k, y_k) + v_k \end{aligned}$$



→ computations with mCRE for each σ-point: ROM + //

→ involves optimal admissible fields for estimation of state

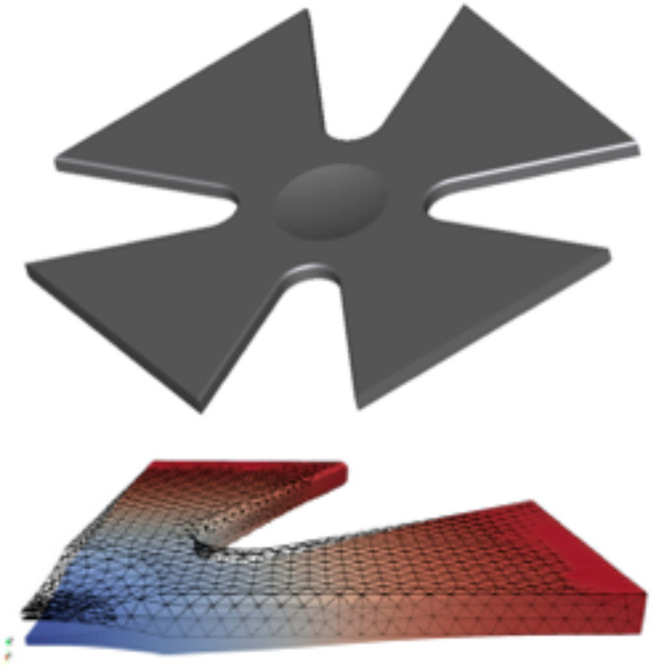
→ automatic calibration of MDKF internal parameters

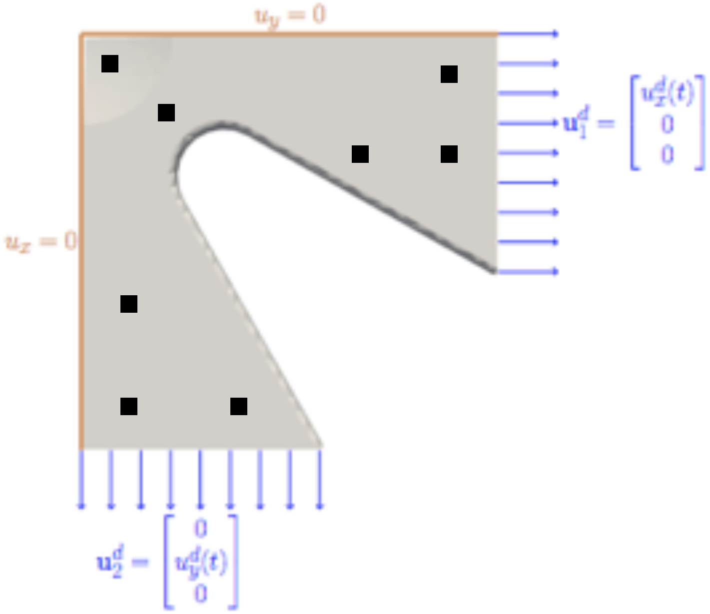
→ enhanced robustness to measurement noise compared to classical UKF

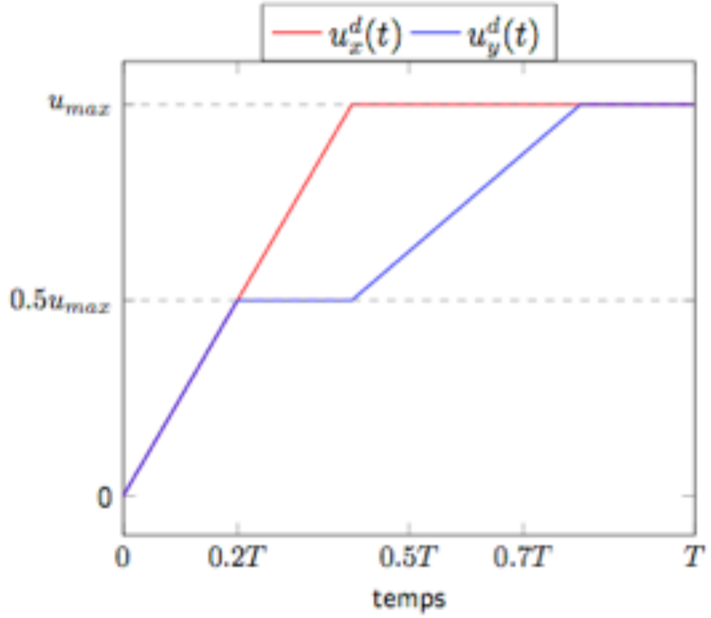
Illustration 1

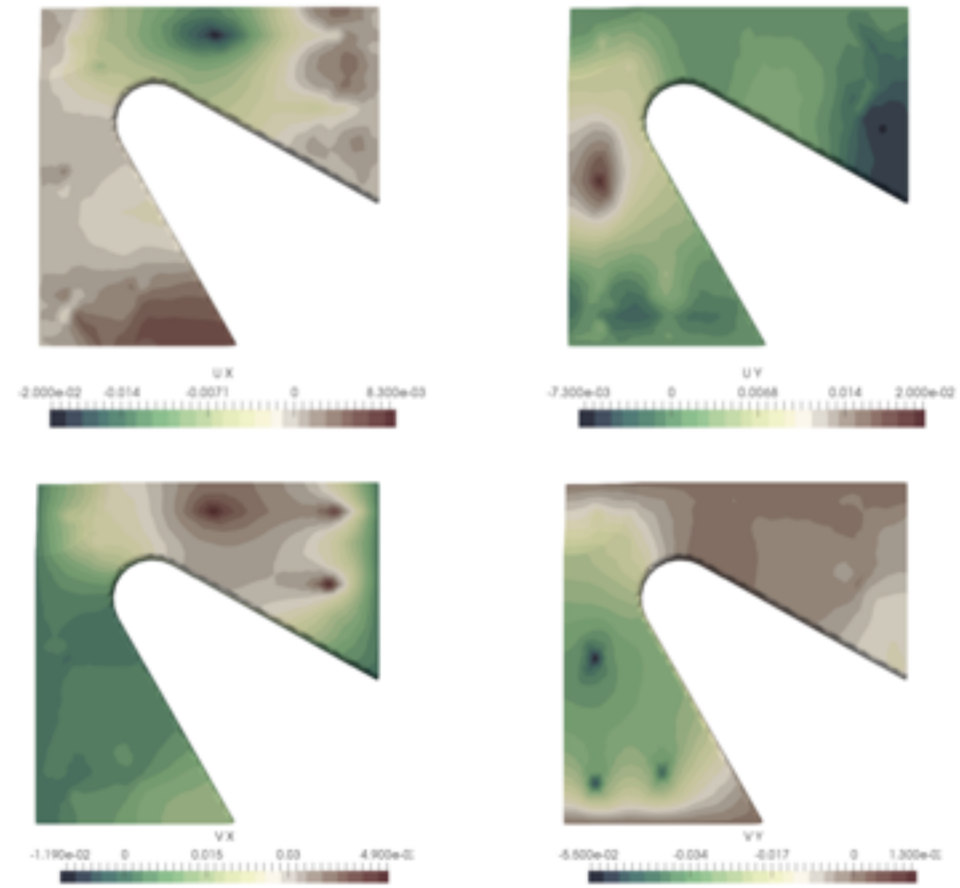
Viscoplasticity case (Prandtl-Reuss)

[Marchand *et al.*]



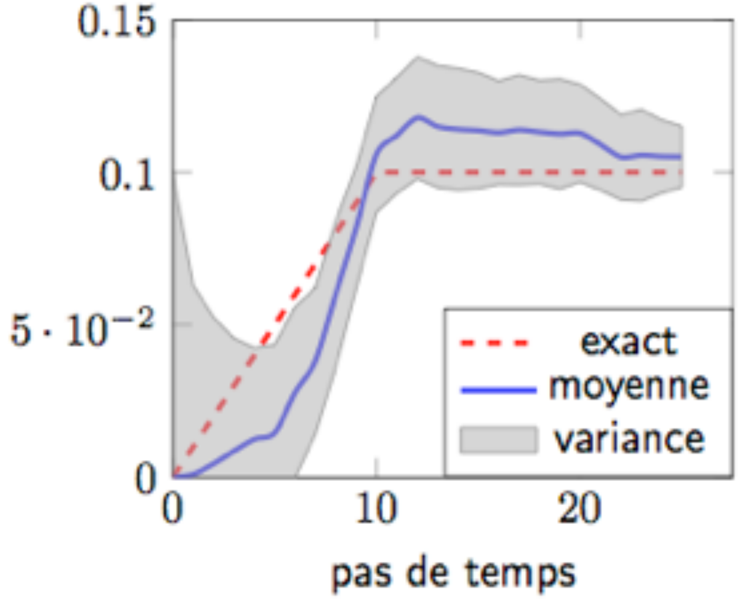


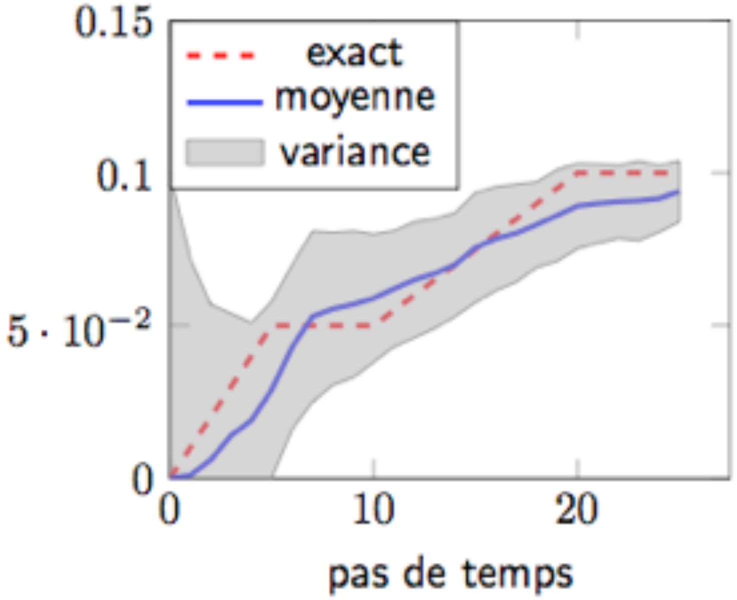




(c) Champ de multiplicateur de Lagrange λ_x

(d) Champ de multiplicateur de Lagrange λ_y





average CPU time $\approx$ 1s

Outline

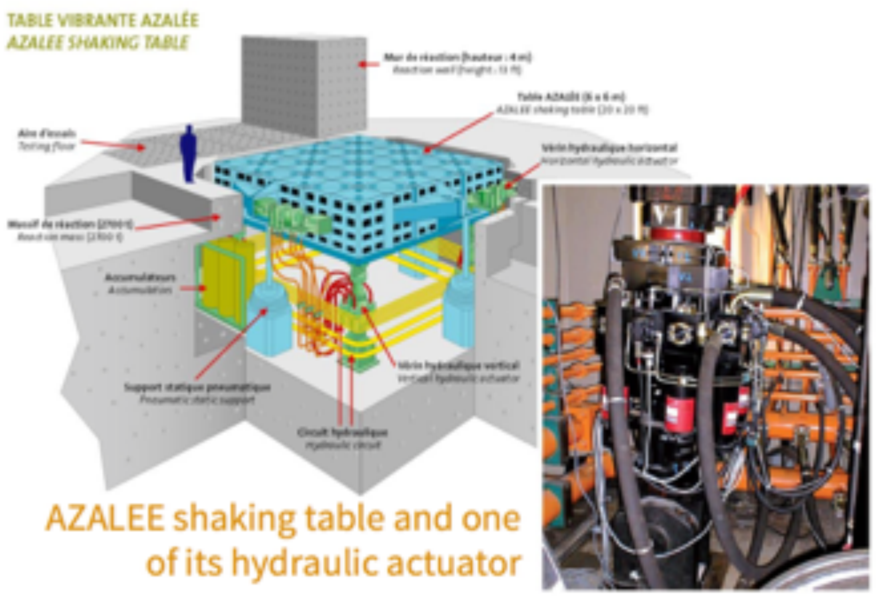
1. Introduction to inverse problems
2. Deterministic inverse problems
3. Stochastic inverse problems
4. Sequential data assimilation
5. Some recent research applications

Application 2

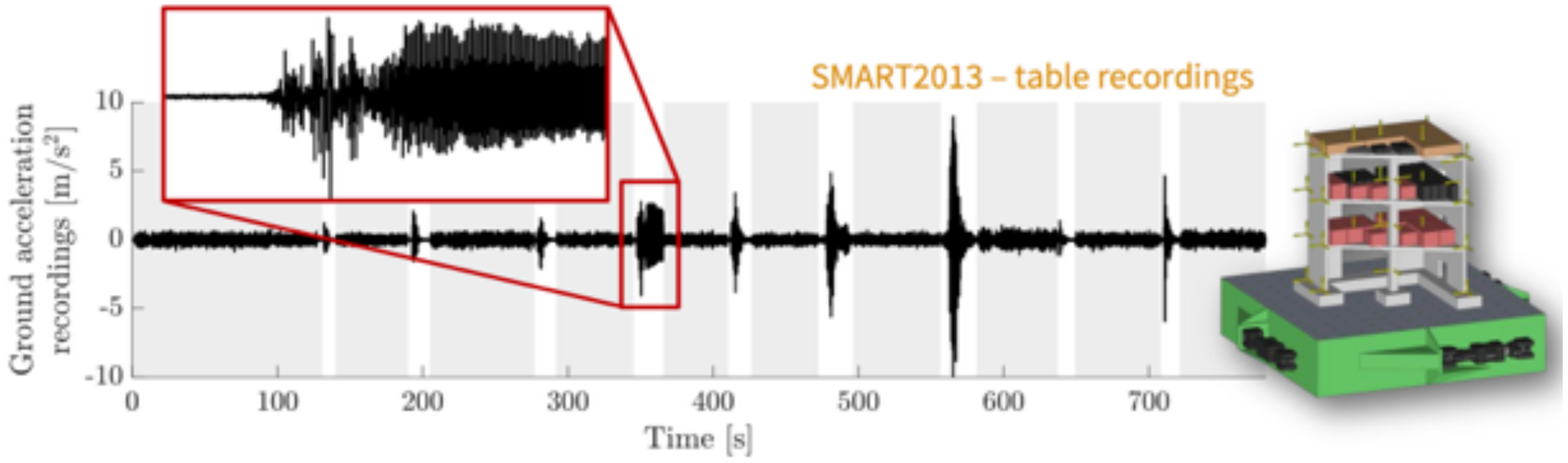
[PhD M. Diaz 2020-2023]

Automated control of shaking-table tests

- low frequency (earthquake engineering)
- SMART 2013 benchmark (CEA/EDF)
- sequence of gradually damaging tests
- If inappropriate control, unstable experiment



AZALEE shaking table and one of its hydraulic actuator

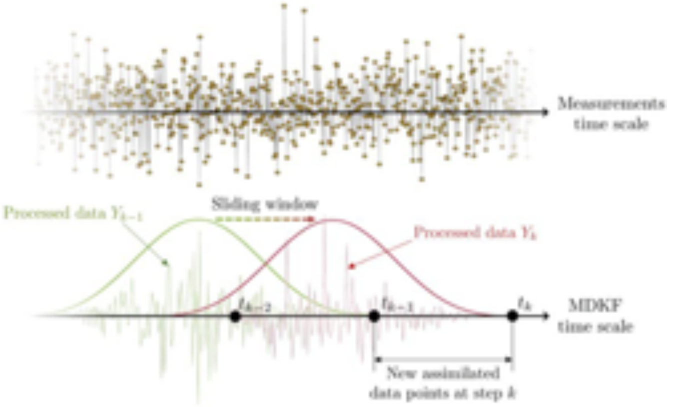


→ modal signature is the key input feature for linear control

→ on the fly assimilation to monitor frequency drop (due to damage)

Application 2

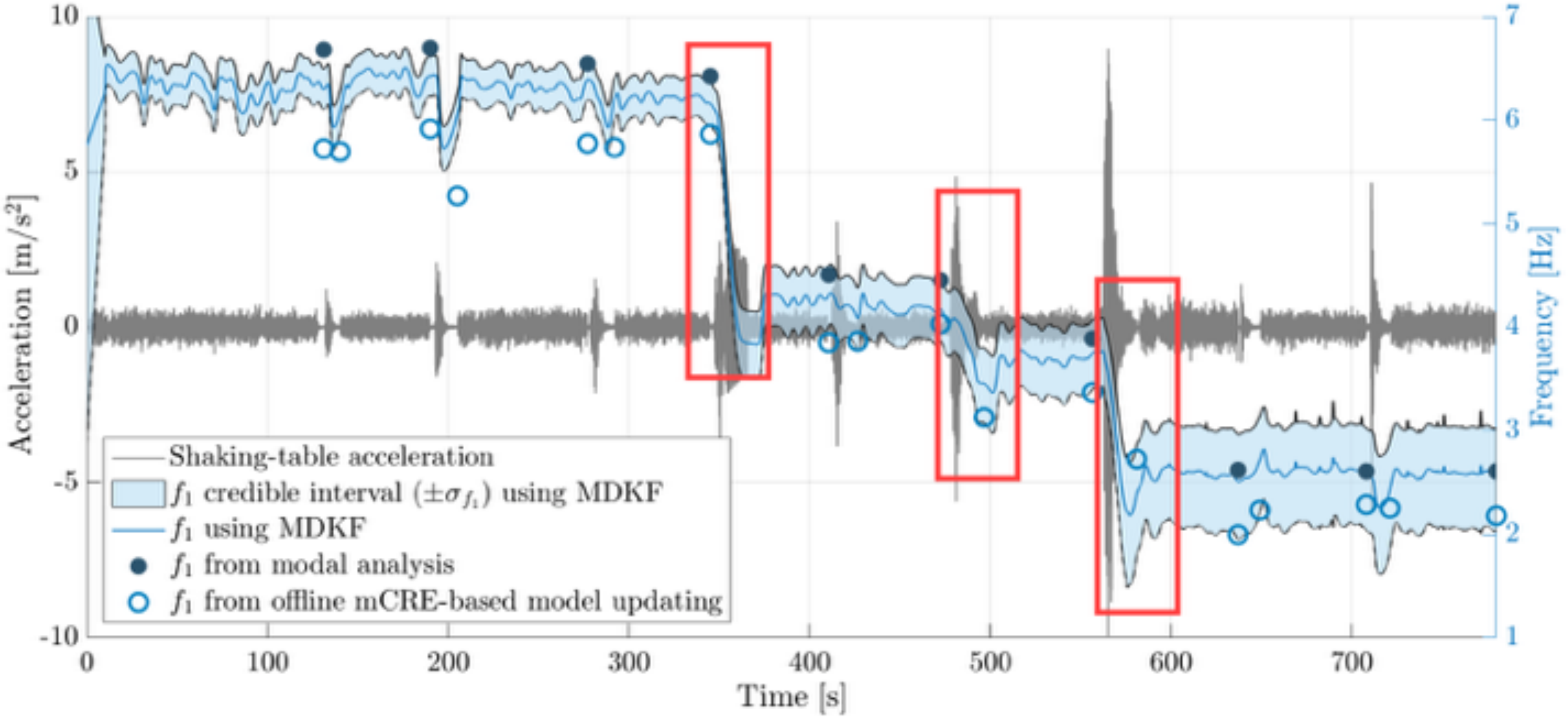
Eigenfrequency tracking from sparse & highly-noisy measurements



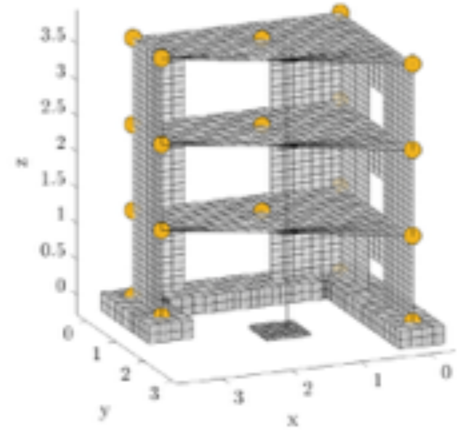
mCRE in the frequency domain

$$e^2_\omega(s,\theta,Y_\omega) = \boxed{\zeta^2_\omega(s,\theta,Y_\omega)} + \alpha\boxed{\|\Pi \circ s - Y_\omega\|^2_G} \Longrightarrow J(\theta) = \int_{D_\omega} \boxed{z(\omega)}\, e^2_\omega(\hat{s}(\theta),\theta,Y_\omega)\, d\omega$$

- results obtained with 41 accelerometer recordings from the campaign
- 1 parameter per wall or floor
- real-time constraint successfully achieved
- tracking of the 3 first eigenfrequencies (first one presented below)



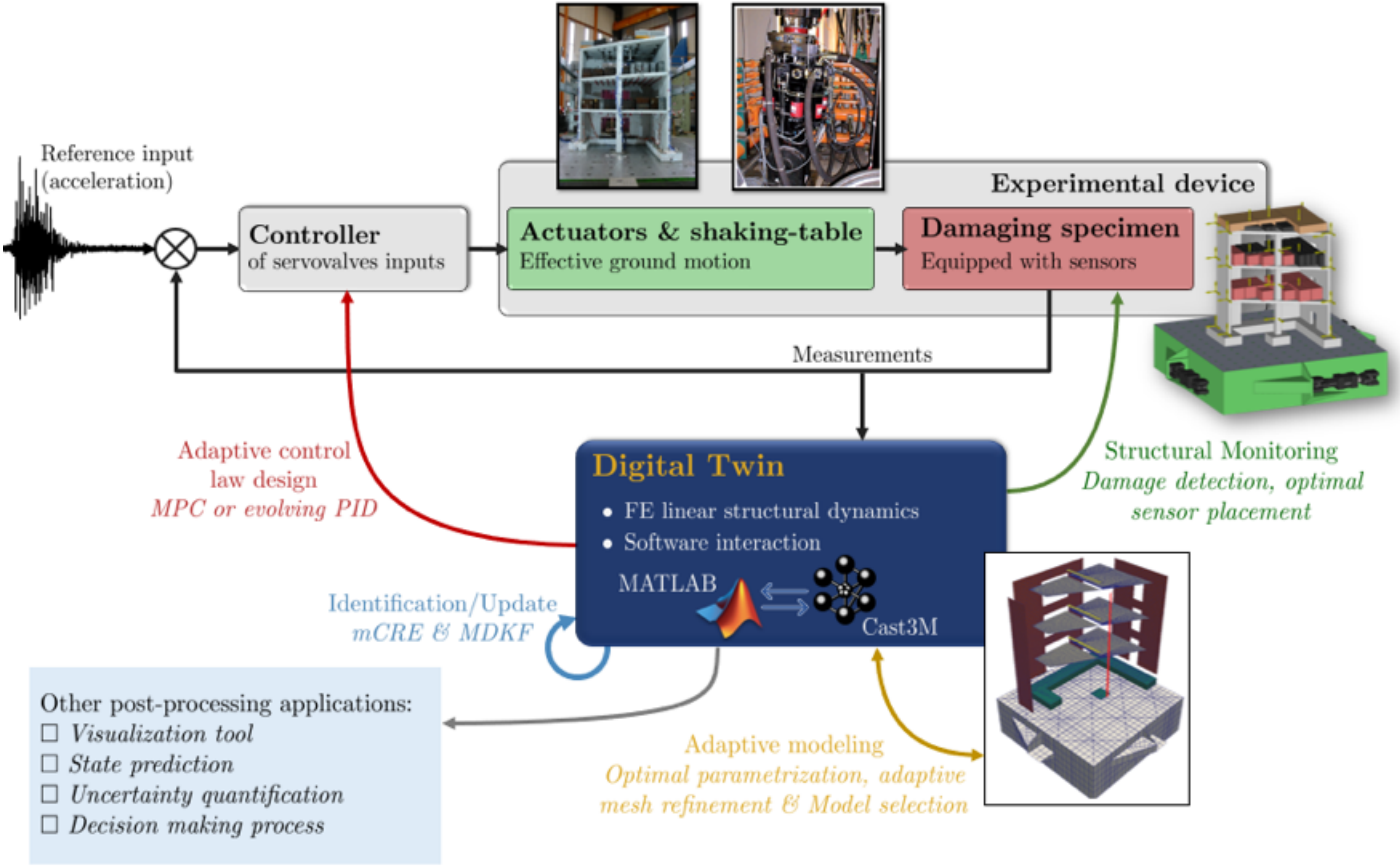
RC specimen of SMART2013 anchored on the AZALEE shaking-table



FE model from CEA-EMSI developed in Cast3M

Illustration 2

[PhD M. Diaz 2020-2023]



Reference input (acceleration)

Controller
of servovalves inputs

Experimental device

Actuators & shaking-table
Effective ground motion

Damaging specimen
Equipped with sensors

Measurements

Adaptive control
law design
MPC or evolving PID

Digital Twin

- FE linear structural dynamics
- Software interaction

MATLAB
Cast3M

Structural Monitoring
Damage detection, optimal sensor placement

Identification/Update
mCRE & MDKF

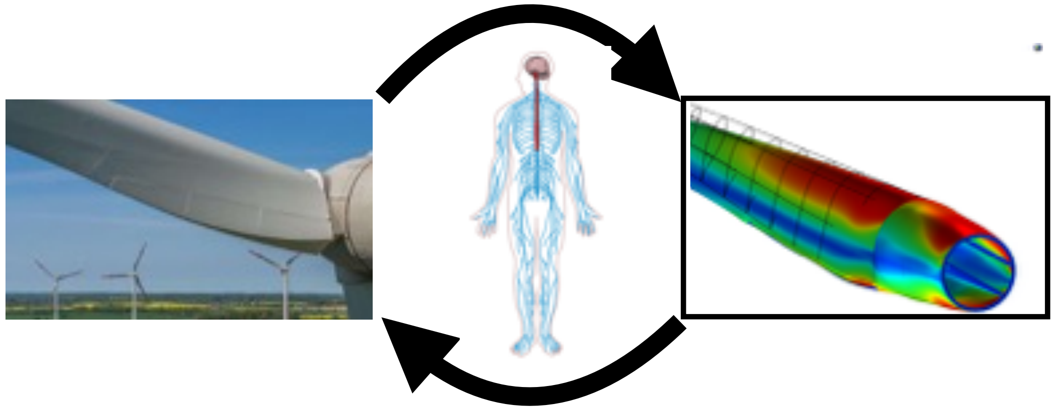
Other post-processing applications:
☐ *Visualization tool*
☐ *State prediction*
☐ *Uncertainty quantification*
☐ *Decision making process*

Adaptive modeling
Optimal parametrization, adaptive mesh refinement & Model selection

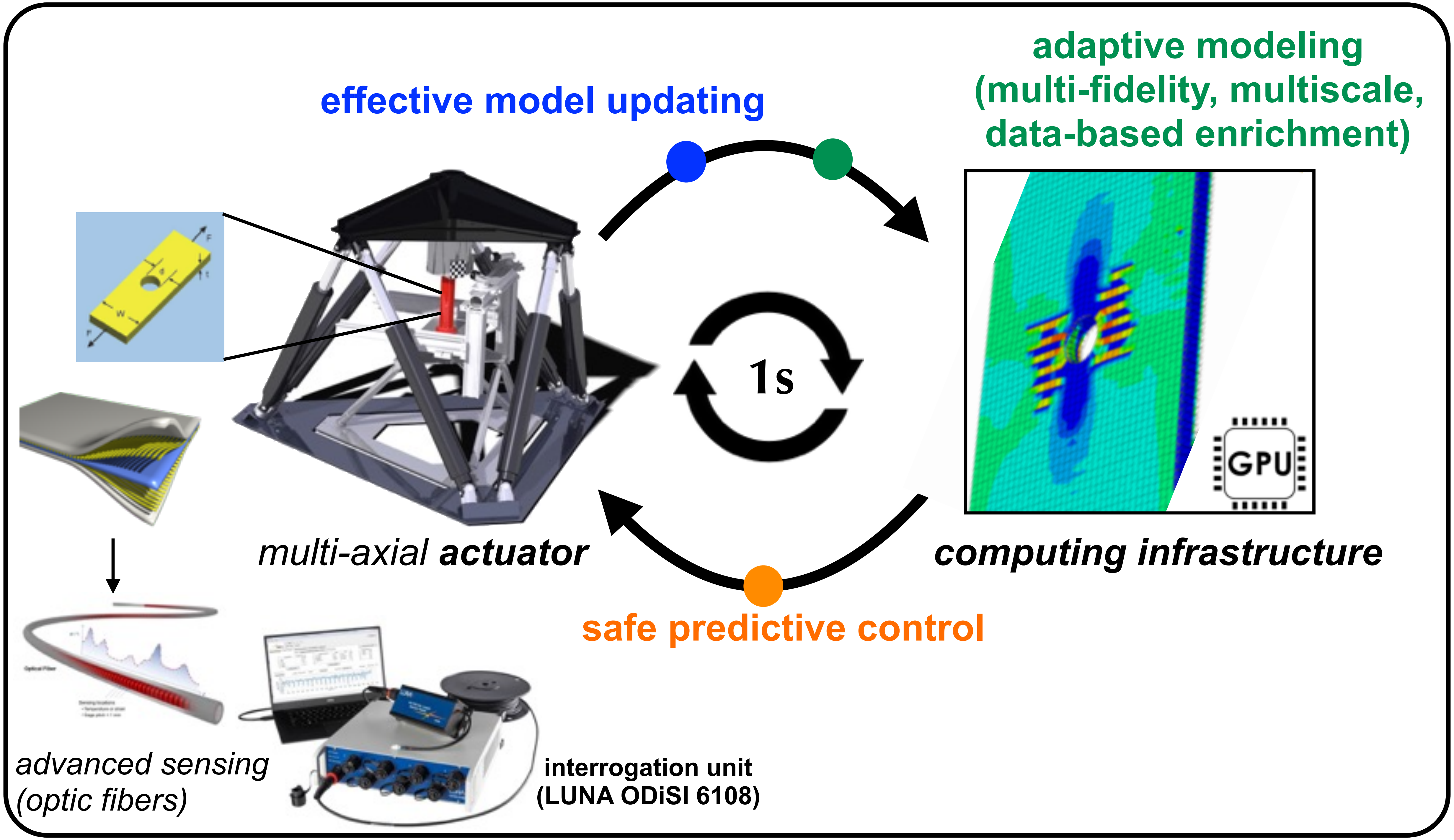
DREAM-ON Project

[ERC-CoG 2021-2026]

European Research Council
Established by the European Commission



SMART ENGINEERING STRUCTURES



effective model updating

adaptive modeling (multi-fidelity, multiscale, data-based enrichment)

multi-axial **actuator**

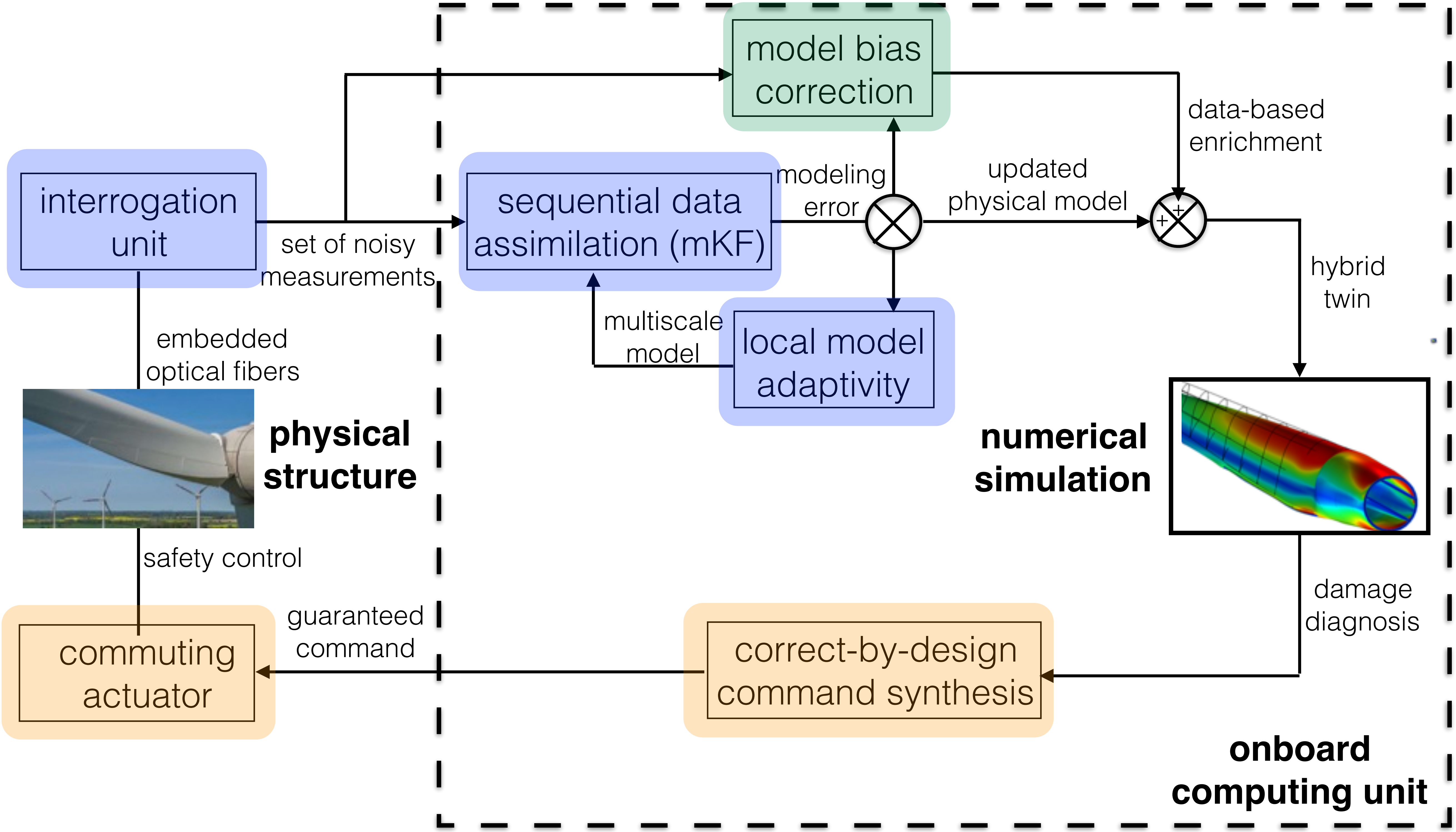
computing infrastructure

safe predictive control

advanced sensing (optic fibers)

interrogation unit (LUNA ODiSI 6108)

DREAM-ON Project



interrogation unit

set of noisy measurements

embedded optical fibers

physical structure

safety control

commuting actuator

guaranteed command

model bias correction

sequential data assimilation (mKF)

modeling error

updated physical model

data-based enrichment

multiscale model

local model adaptivity

hybrid twin

numerical simulation

damage diagnosis

correct-by-design command synthesis

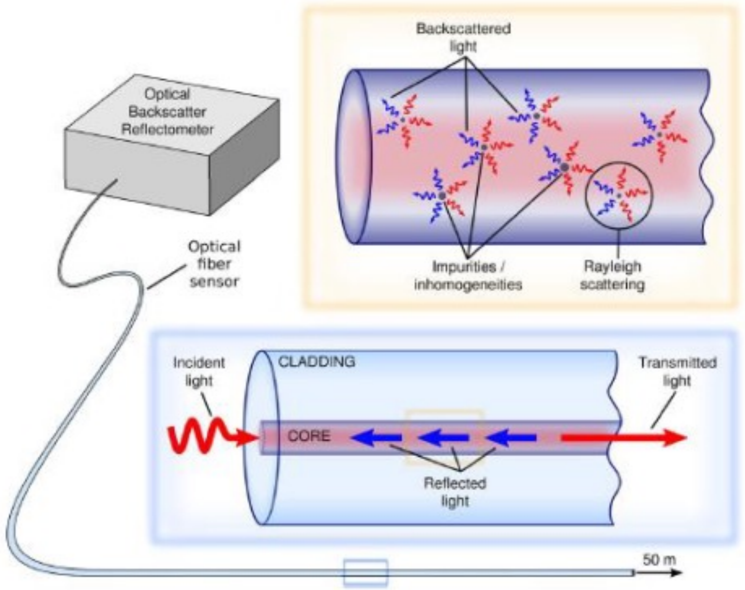
onboard computing unit

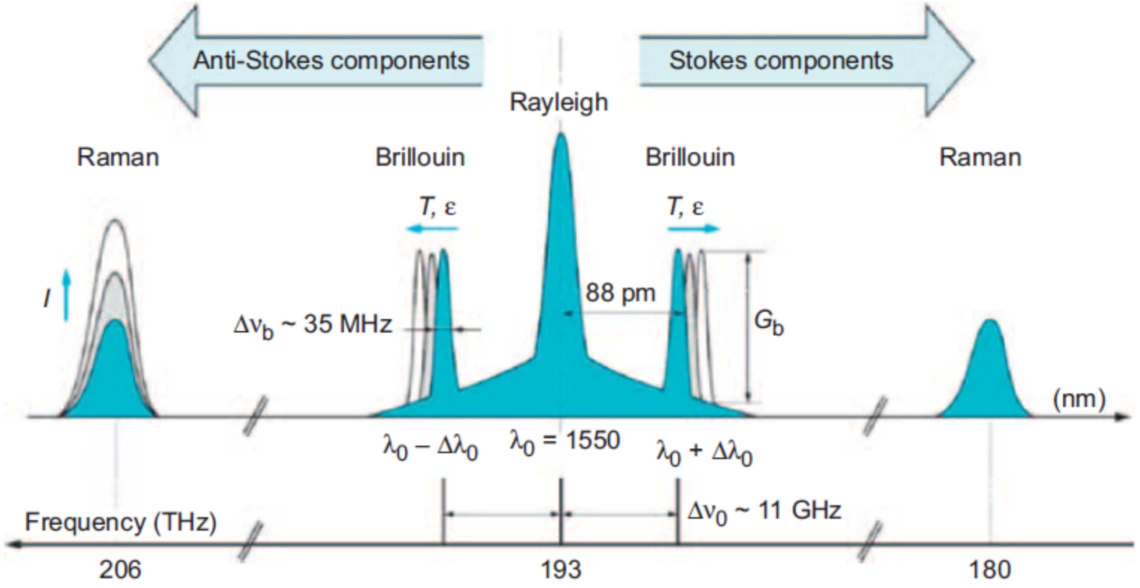
mDKF Strategy with Optic Fiber Sensing

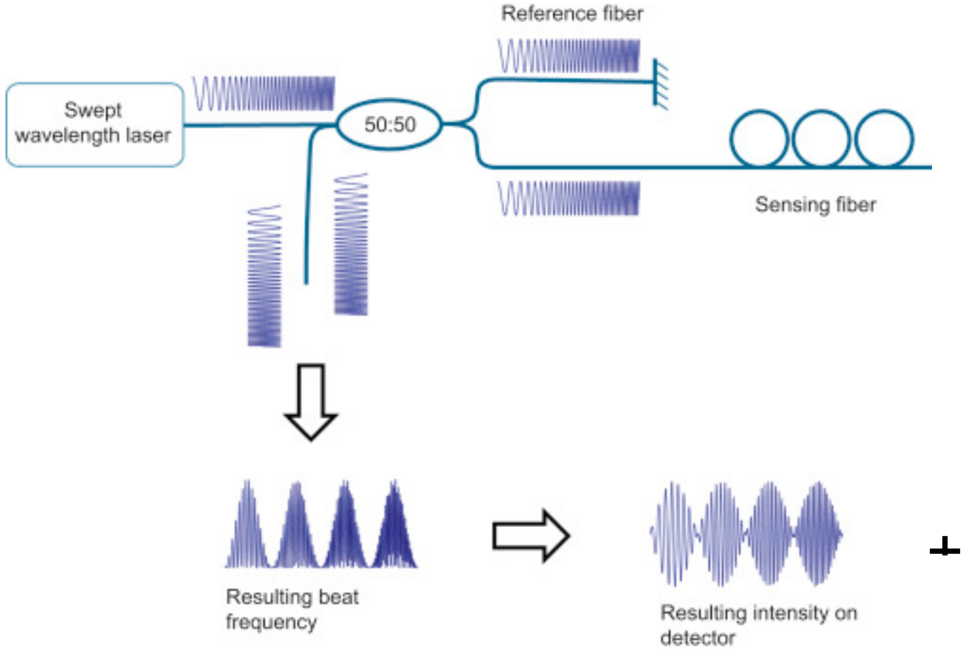
[PhD S. Farahbakhsh 2021-2024]

Distributed Optic Fiber Sensing (DOFS)

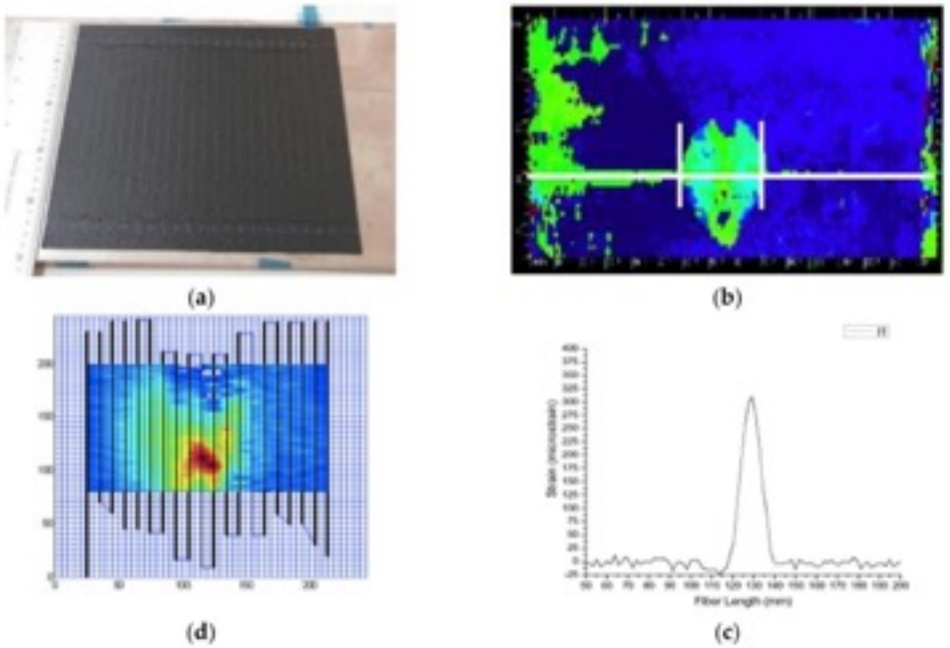
→ *Rayleigh backscattering + OFDR technique*







+cross-

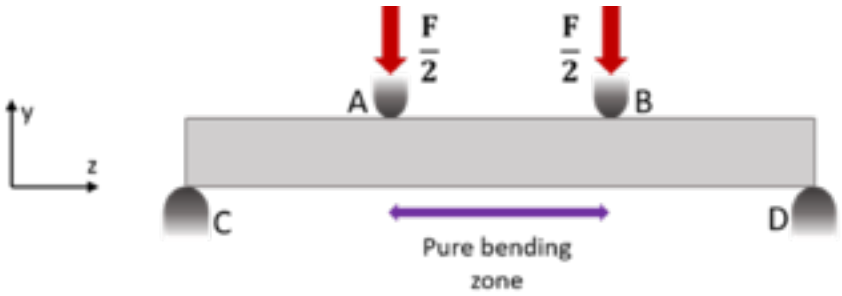


L.C., S. Farahbakhsh, M. Poncelet, An educational review on distributed optic fiber sensing based on Rayleigh backscattering for damage tracking and SHM, *Measurement Science & Technology*, 33:124008 (2022)

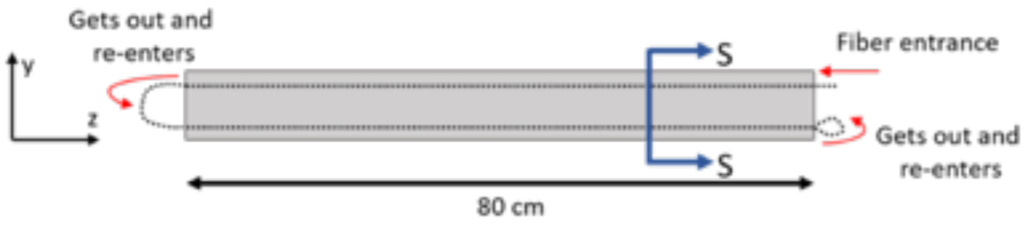
mDKF Strategy with Optic Fiber Sensing

[PhD S. Farahbakhsh 2021-2024]

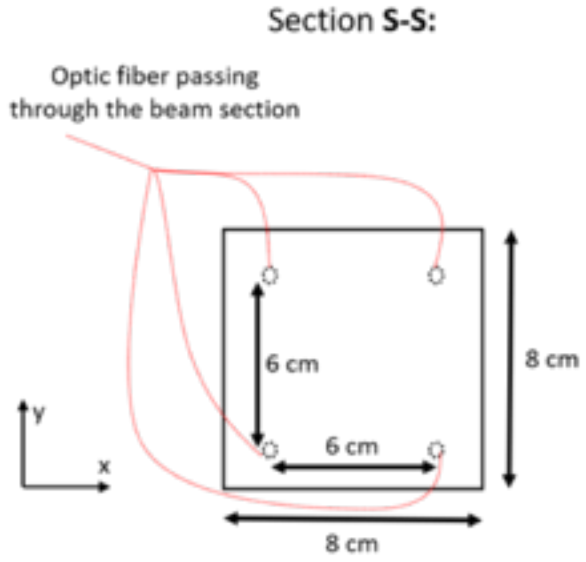
Some 1st experiments



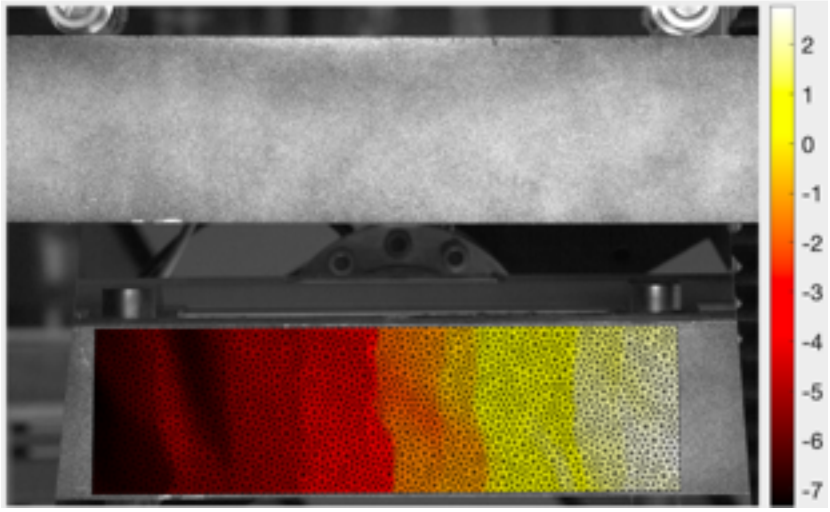
(a)



(b)

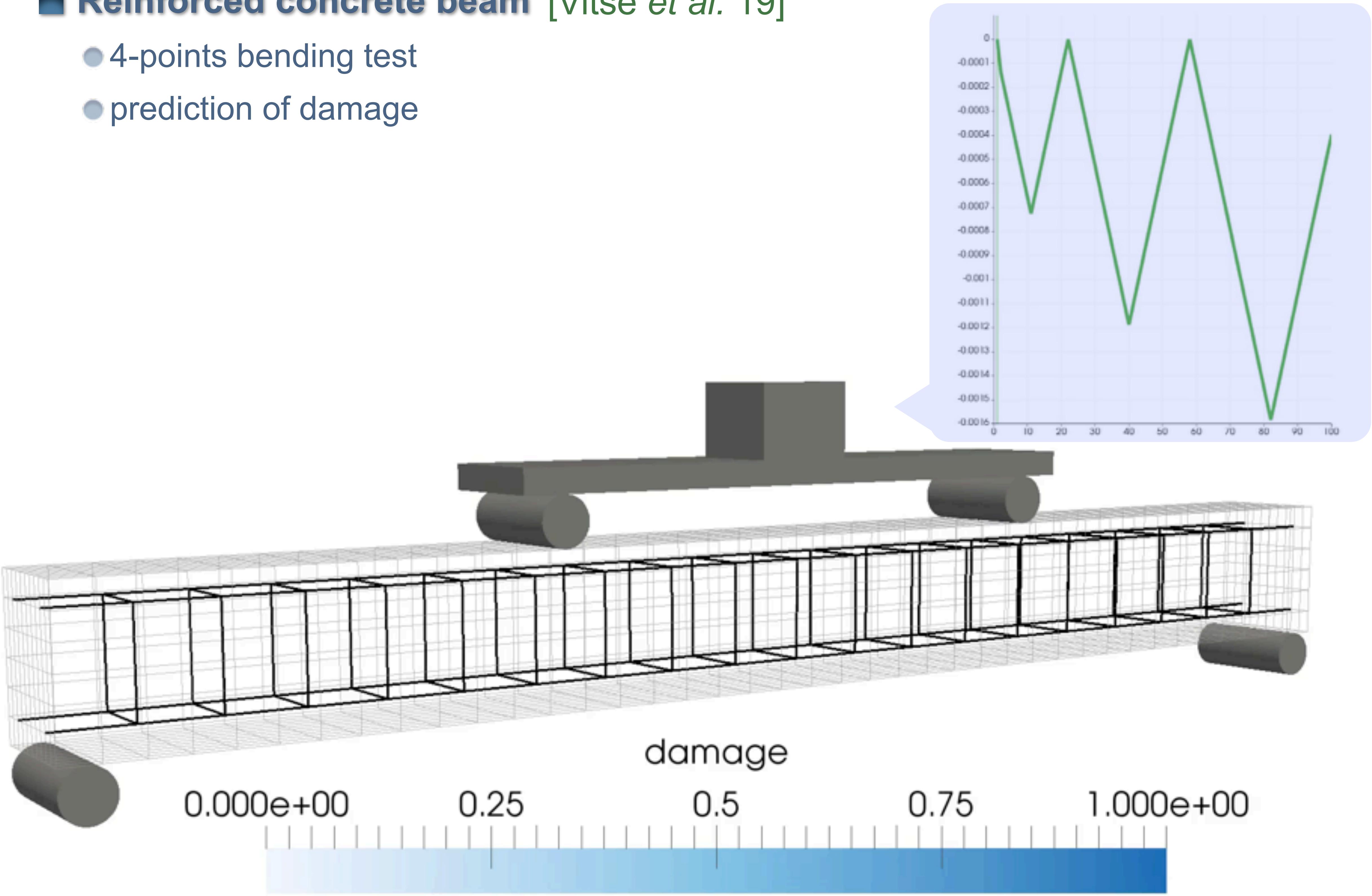


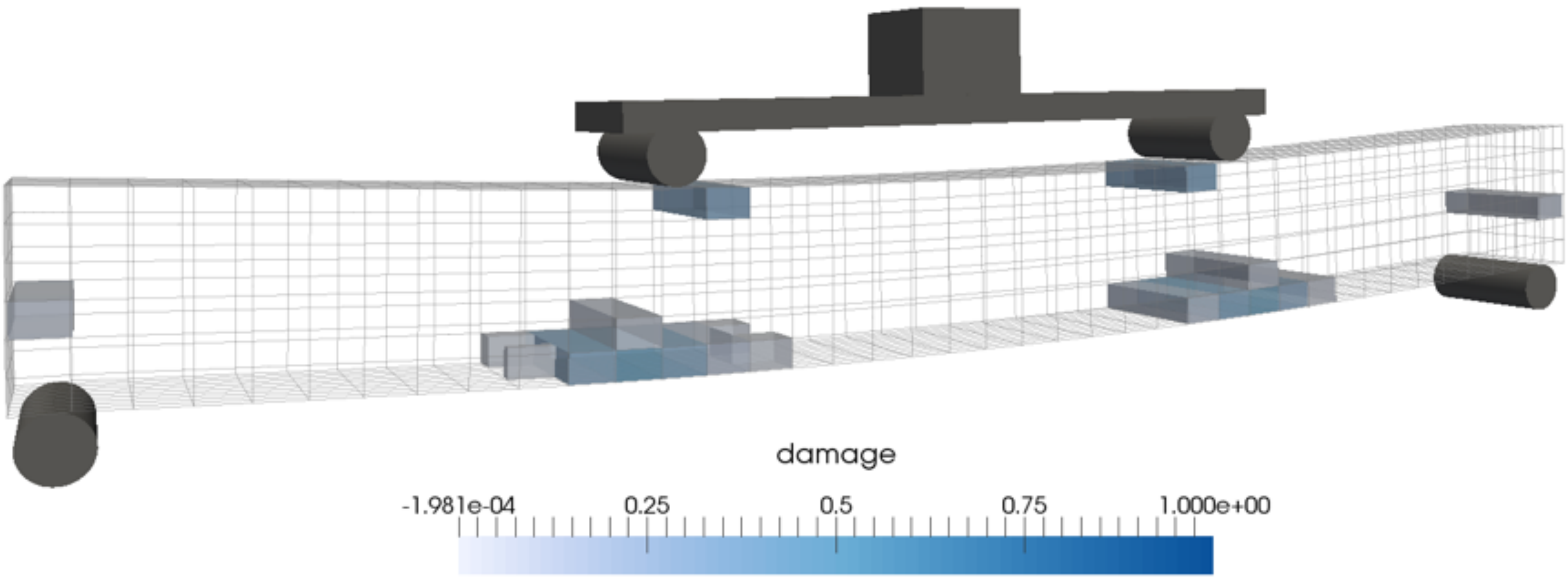
(c)

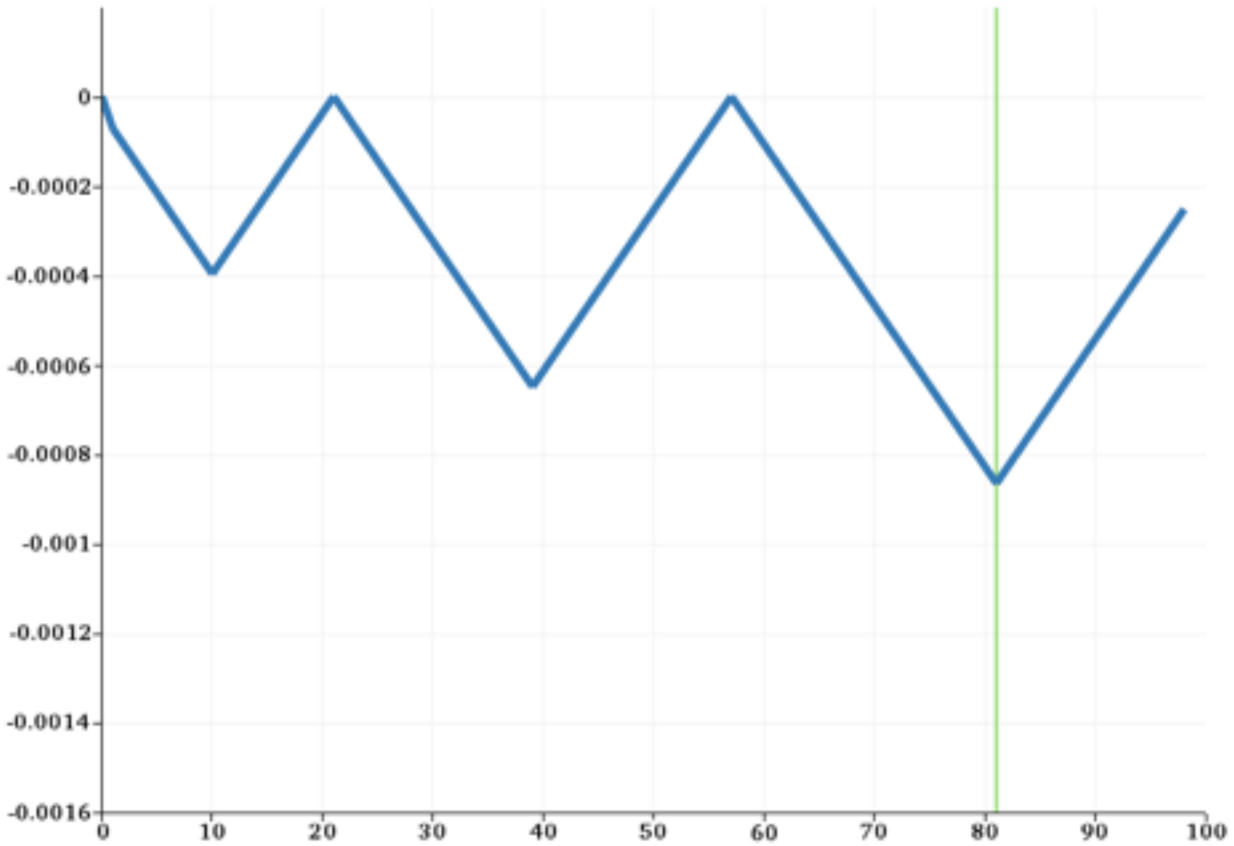


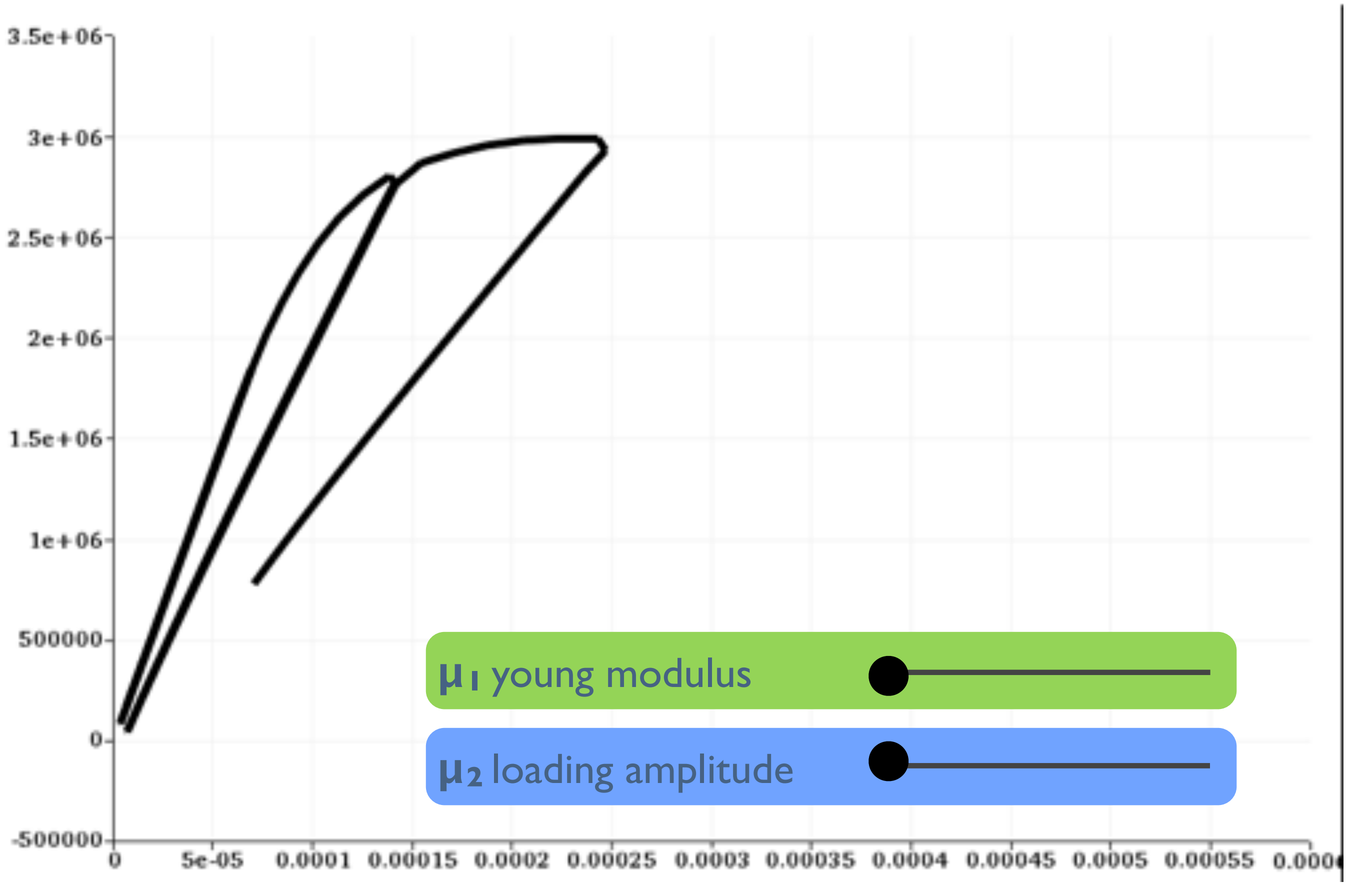
Example: 4-points Bending Test

- **Reinforced concrete beam** [Vitse *et al.* 19]
  - 4-points bending test
  - prediction of damage



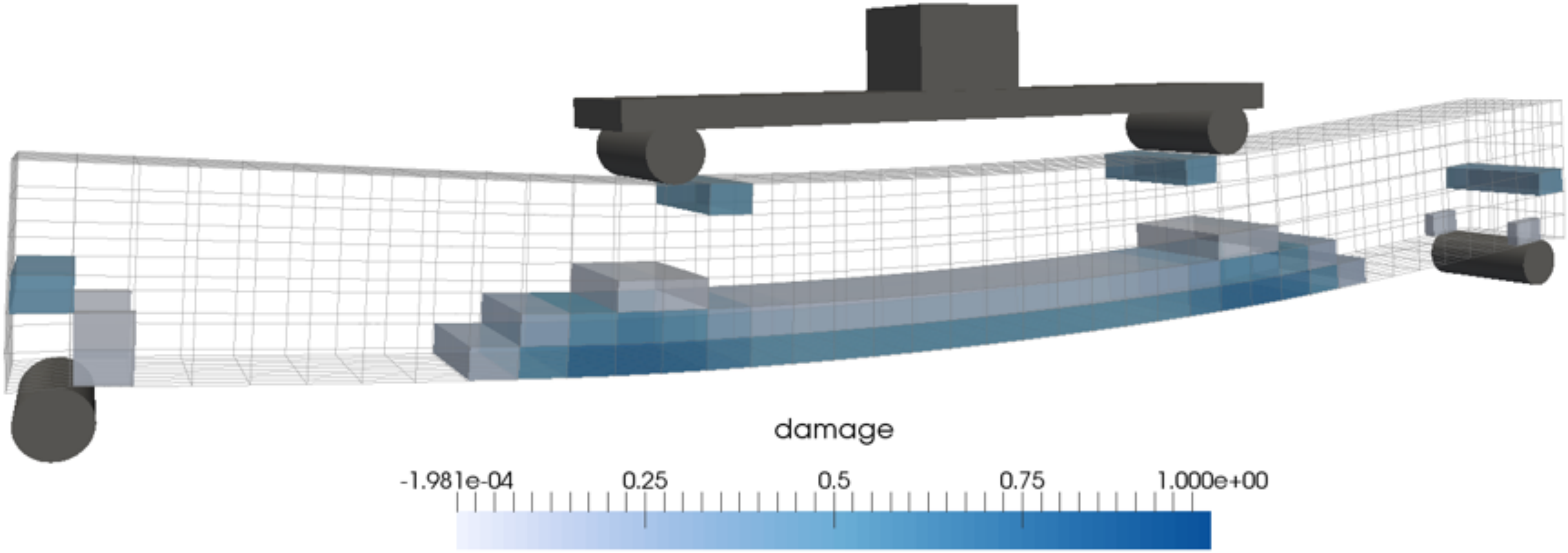


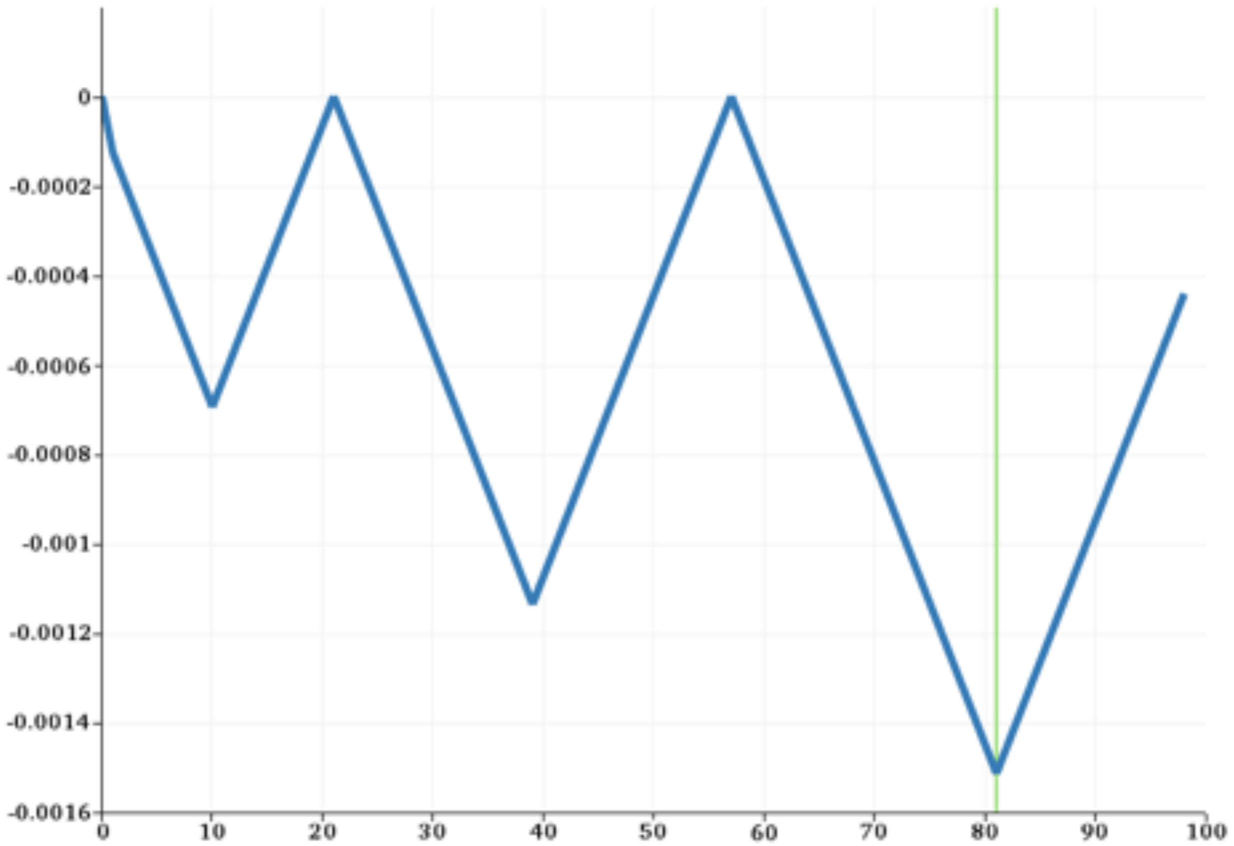


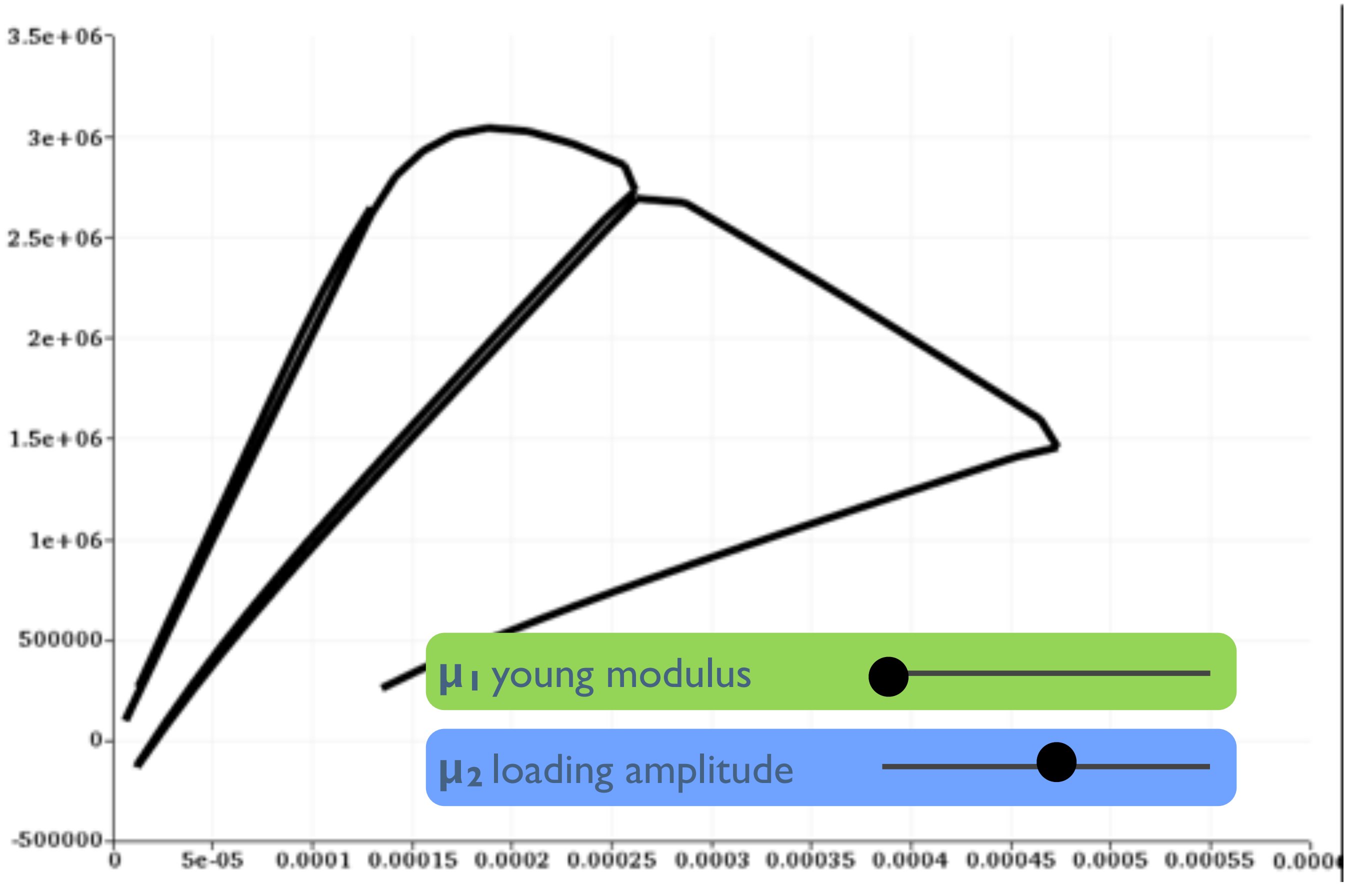


PXDMF Reader

A Separated Variable Reader for ParaView

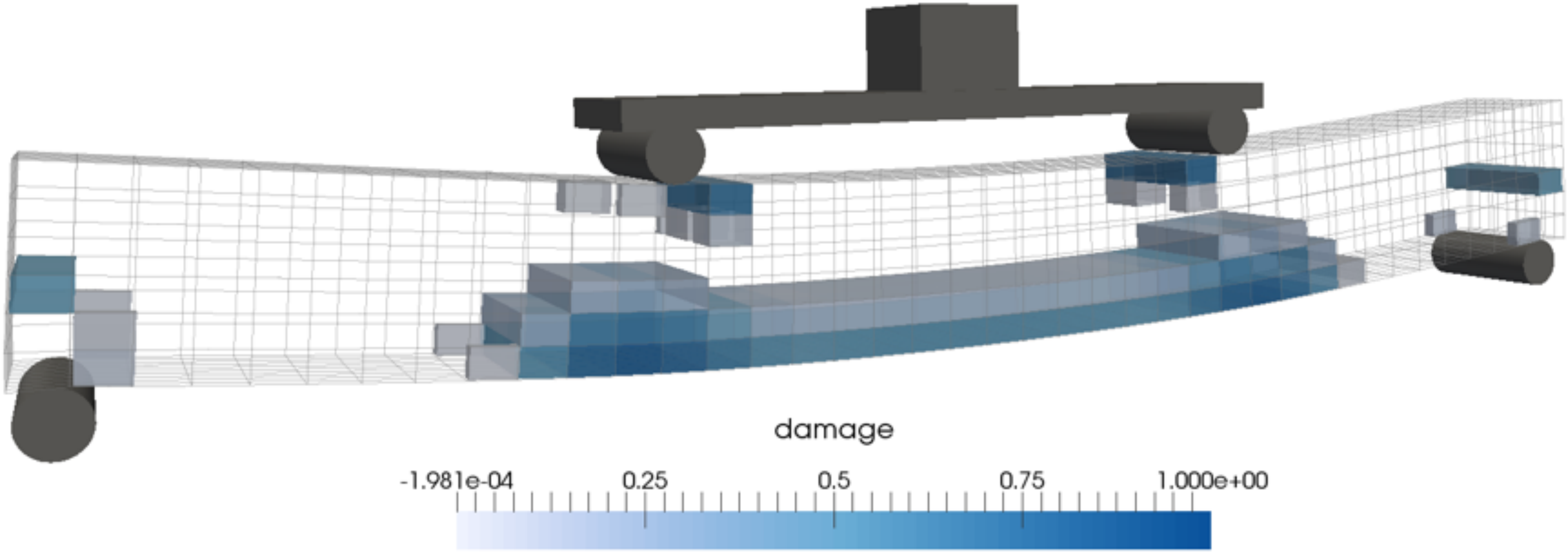


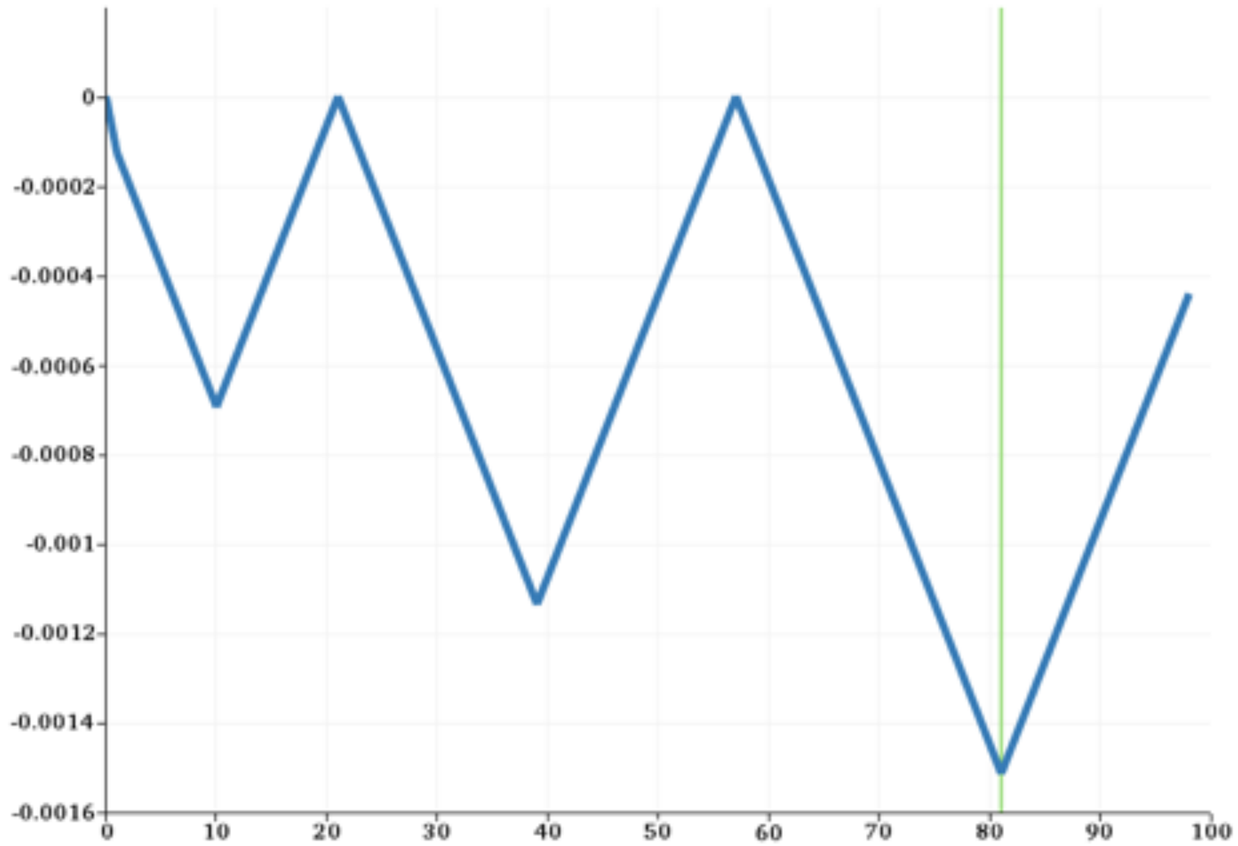


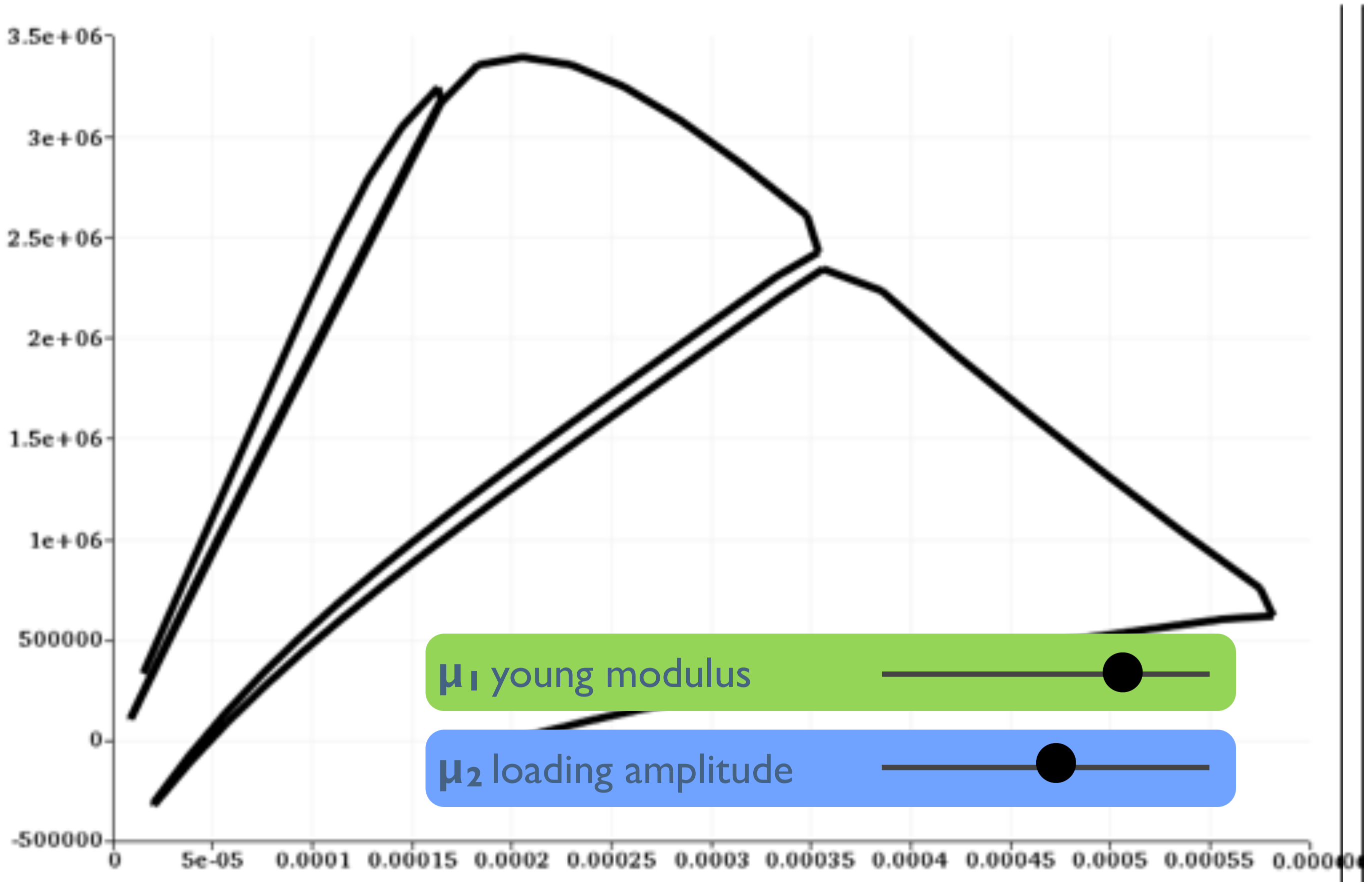


PXDMF Reader

A Separated Variable Reader for ParaView







PXDMF Reader

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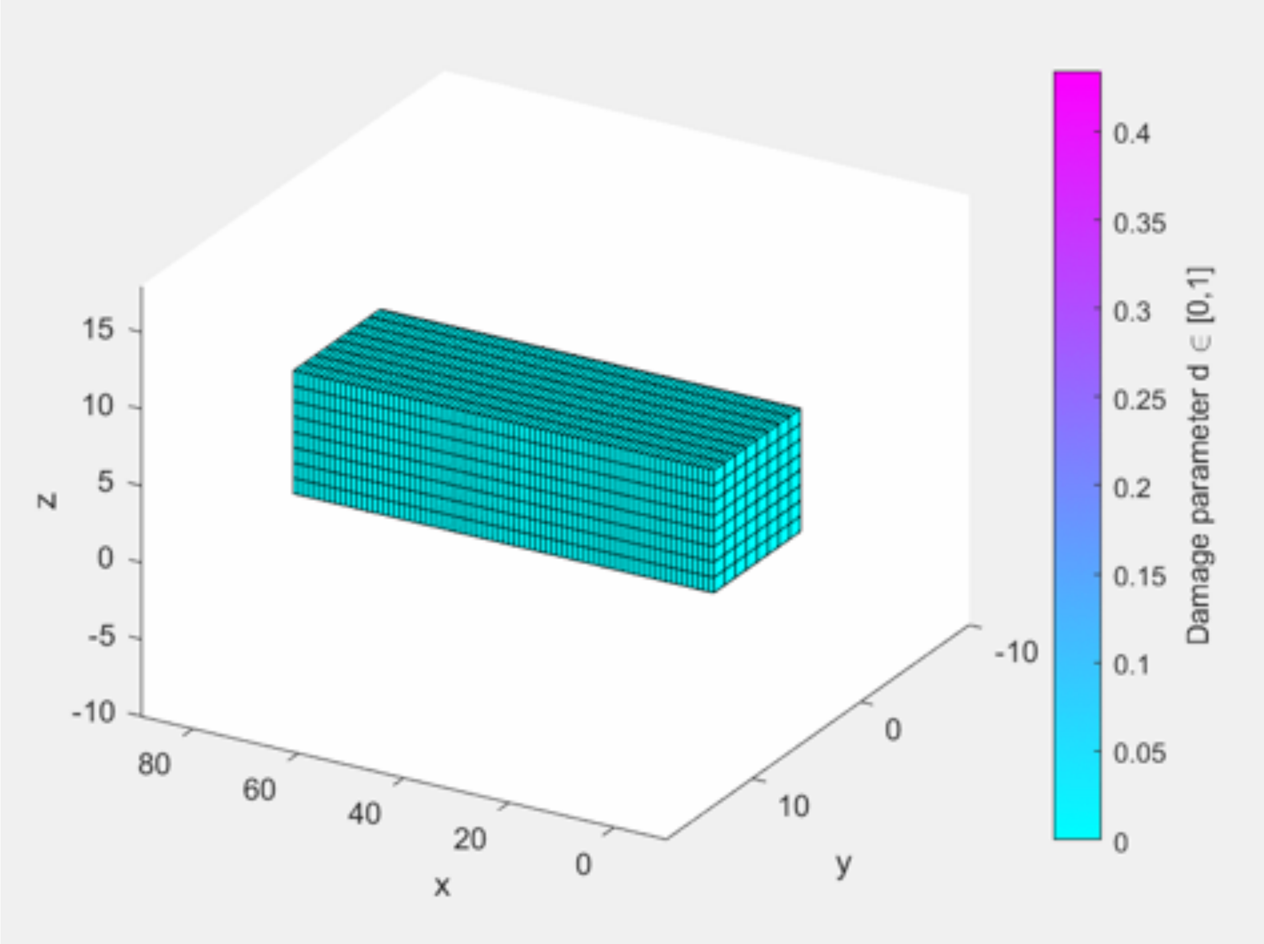
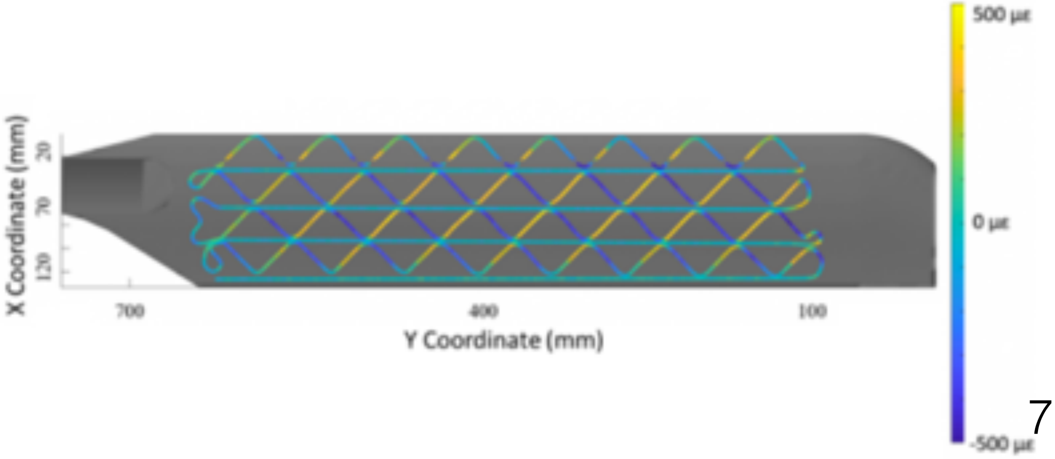
mDKF Strategy with Optic Fiber Sensing

- **Results** [Farahbakhsh *et al. 23*]
  - Influence of the fibers radio, of the resolution (adaptive model adaptivity through CRE)



  - Optimal fiber placement using Entropy of Information [Chamoin *et al. 23*]

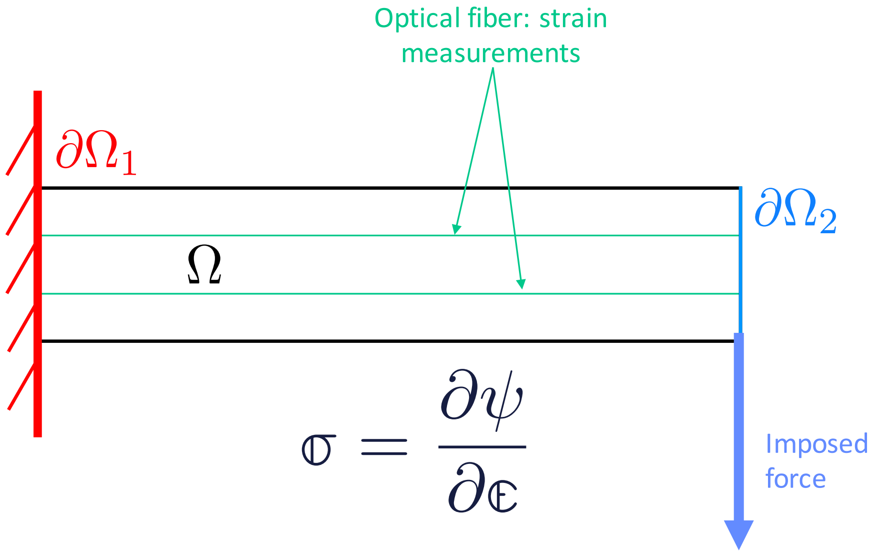


Physics-Augmented NN for Model Enrichment

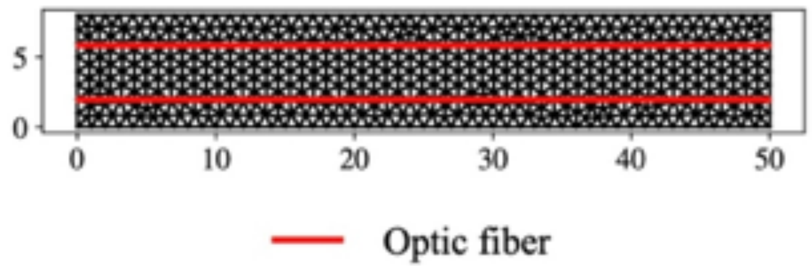
[PhD A. Benady 2021-2024]

General strategy

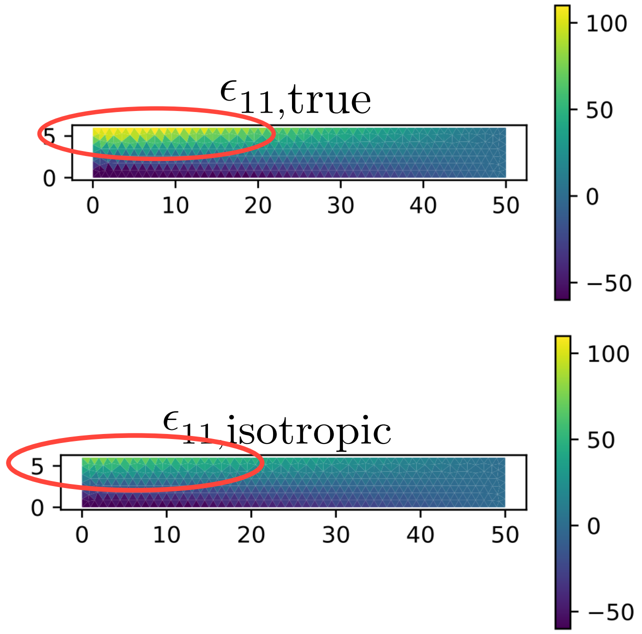
- unsupervised strategy to correct bias in the constitutive relations
- thermodynamics-consistent architecture, with mCRE as loss function
- initialization with a simple law
- minimization from gradient descent (adjoint state method)
- modeling error information over the whole structure (not only at measurement points)
- automatic tuning of hyper parameters (batch size, number of epochs, learning rate…)

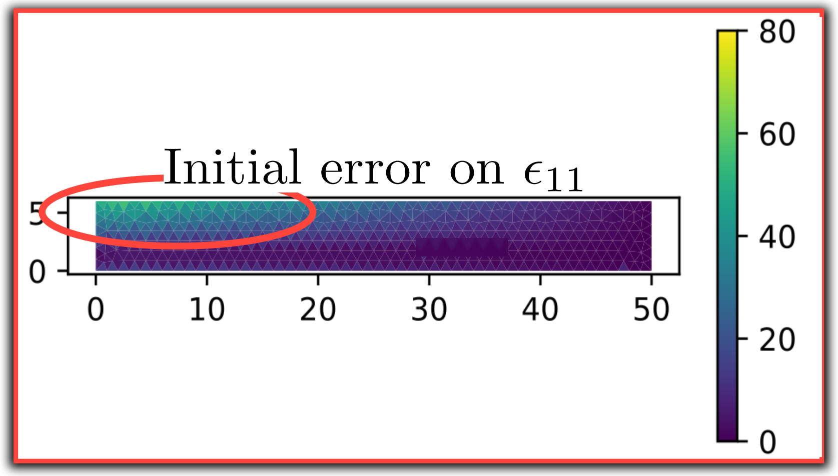


$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}$$



- Truth: $\psi(\boldsymbol{\varepsilon}) = \frac{1}{2}E_1(1-d_1) < \epsilon_{11} >_+^2 + \frac{1}{2}E_1 < \epsilon_{11} >_-^2 + \frac{1}{2}E_2\,\epsilon_{22}^2 + G\,\epsilon_{21}^2$
- Initial guess: $\psi(\boldsymbol{\varepsilon}) = \frac{1}{2}E_1\,\epsilon_{11}^2 + \frac{1}{2}E_2\,\epsilon_{22}^2 + G\,\epsilon_{21}^2$





Physics-Augmented NN for Model Enrichment

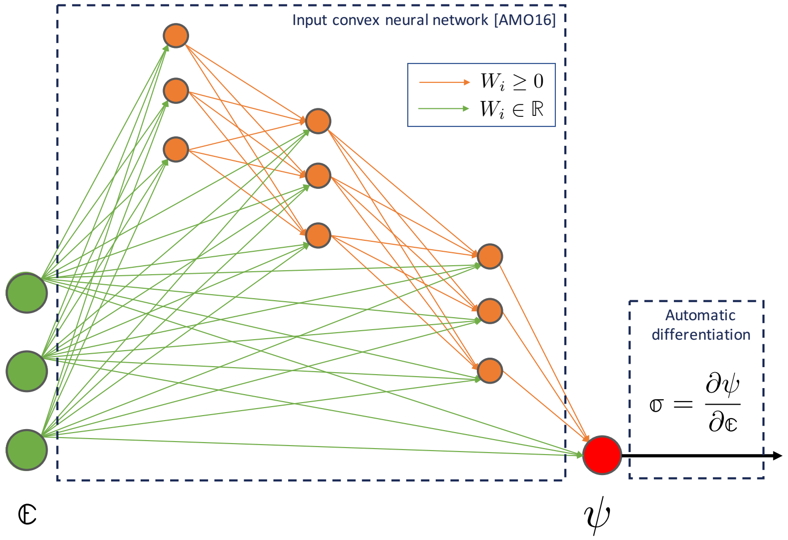
[PhD A. Benady 2021-2024]

Some 1st results [Benady *et al. 23*]

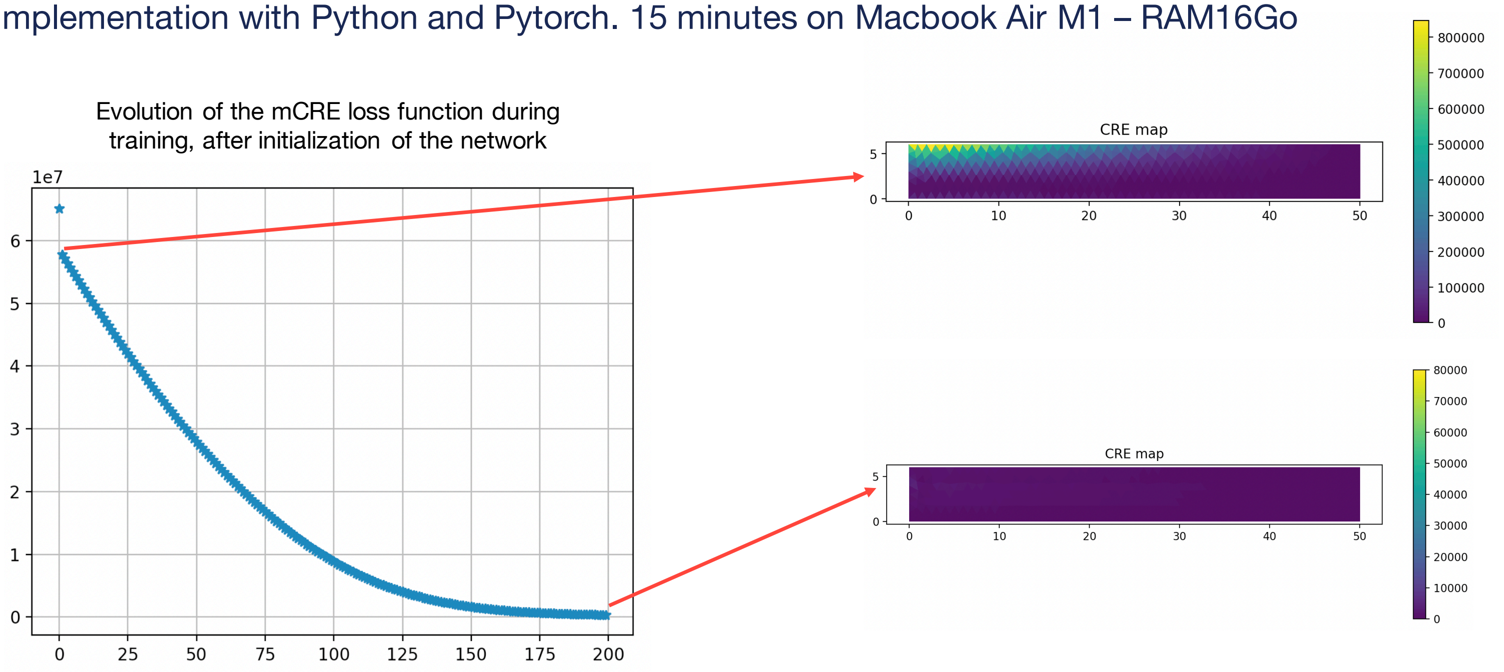
- Loss function: $$\mathcal{E}^2_{mCRE}(\hat{\mathbf{u}},\hat{\sigma}) = \underbrace{\boxed{||\Pi\varepsilon(\hat{\mathbf{u}}) - \varepsilon_{obs}||}^2}_{\text{distance to observations}} + \underbrace{\boxed{\int_\Omega \psi(\hat{\mathbf{u}}) + \psi^*(\hat{\sigma}) - \hat{\sigma} : \varepsilon(\hat{\mathbf{u}})}}_{\text{modeling error term}}$$

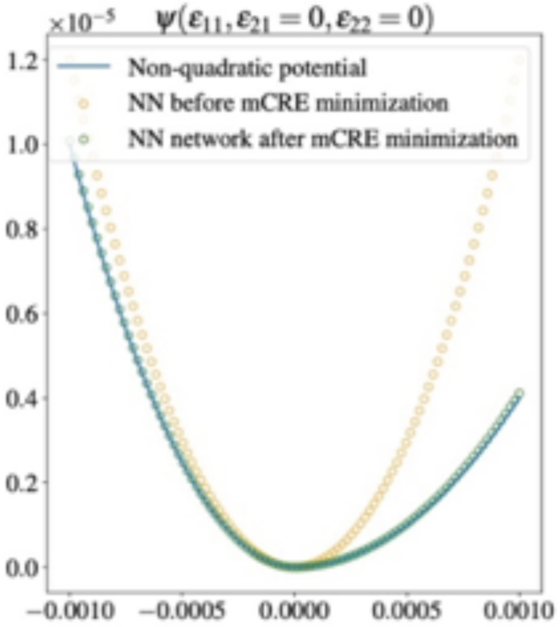
- Specific architecture ensuring convexity of potentials

B. Amos, L. Xu, JZ. Kolter, Input Convex Neural Networks, *https://arxiv.org/abs/1609.07152* (2016)



- Implementation with Python and Pytorch. 15 minutes on Macbook Air M1 – RAM16Go





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