From granular media to suspensions.

Taking contacts and close interactions into account.

XX Jacques-Louis Lions Spanish French School
On Numerical Simulations in Physics and Engineering

Barcelona, Spain, 3 - 7 July 2023

Aline Lefebvre-Lepot

Granular media and suspensions.

Macroscopic, non-brownian grains
Dry system or Stokes flow

Rheology.

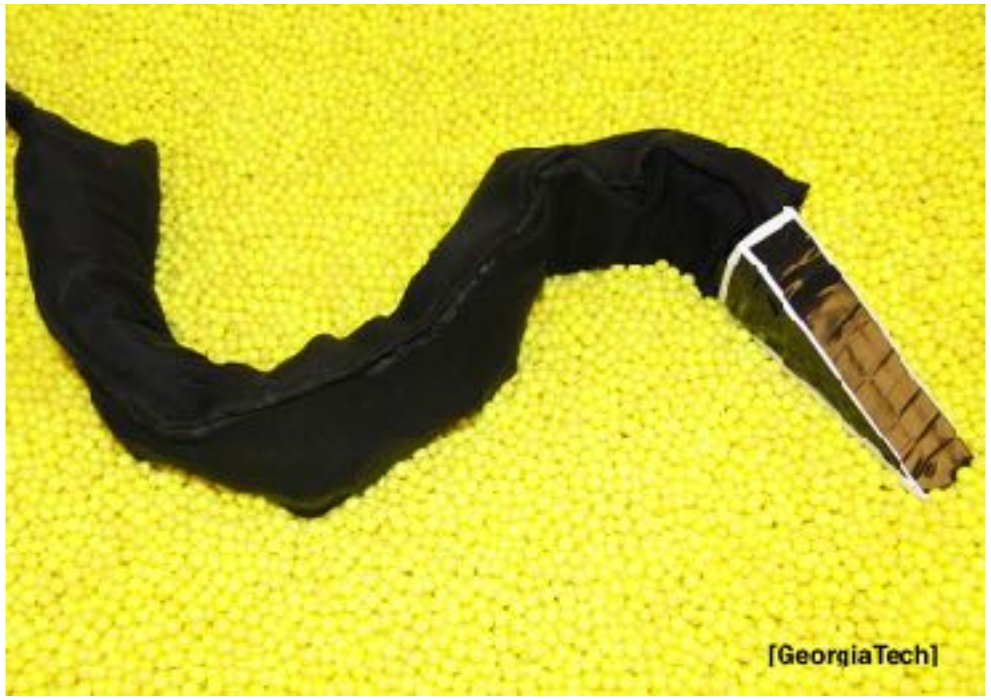
Rheology : ” the branch of physics concerned with the flow and change of shape of matter ”

[Collins English Dictionary]

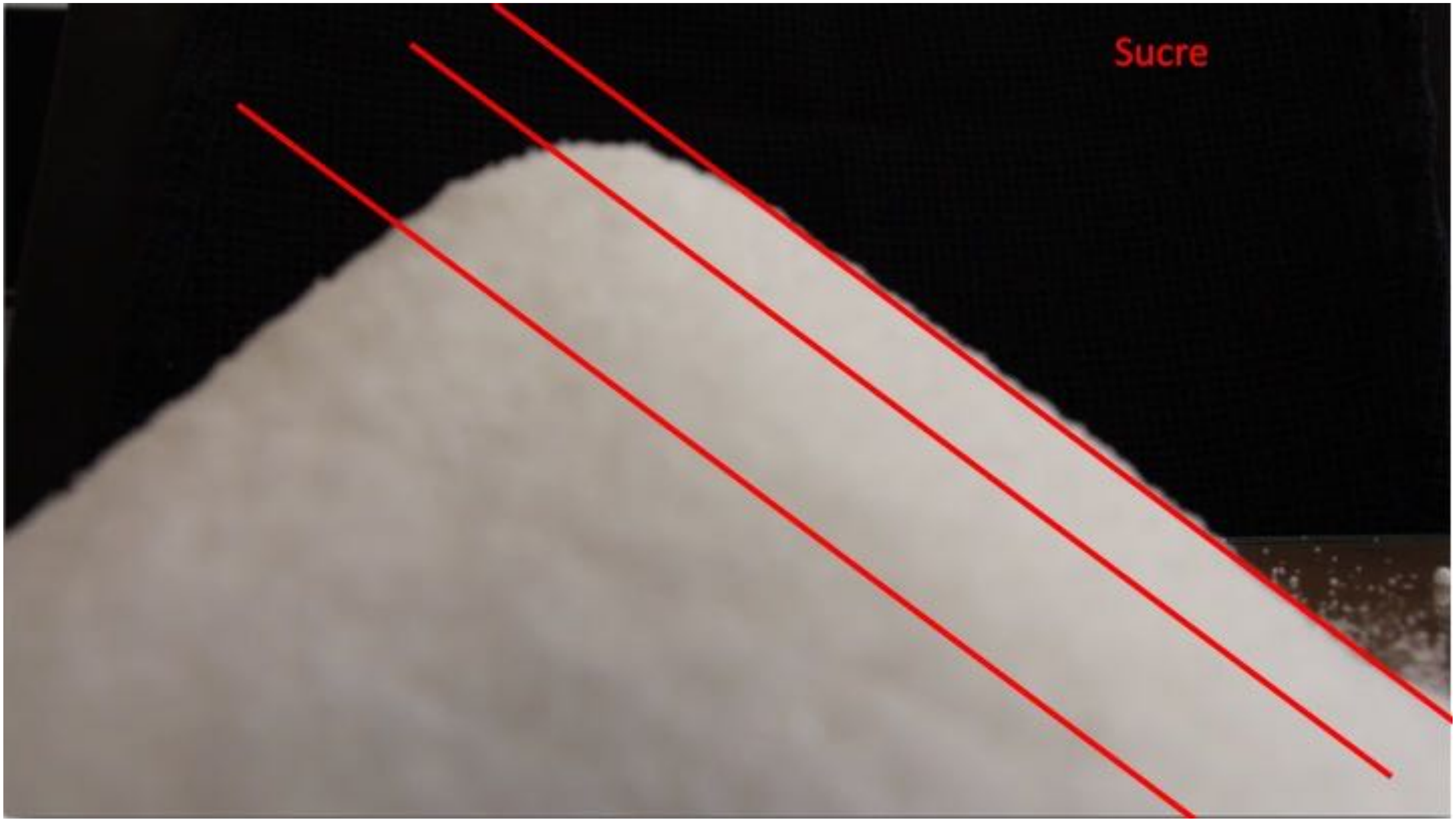
- flow, segregation, mixing, blocking, collapse...

Macroscopic behaviour

Learning to swim?



A complex behavior: kitchen experiment…



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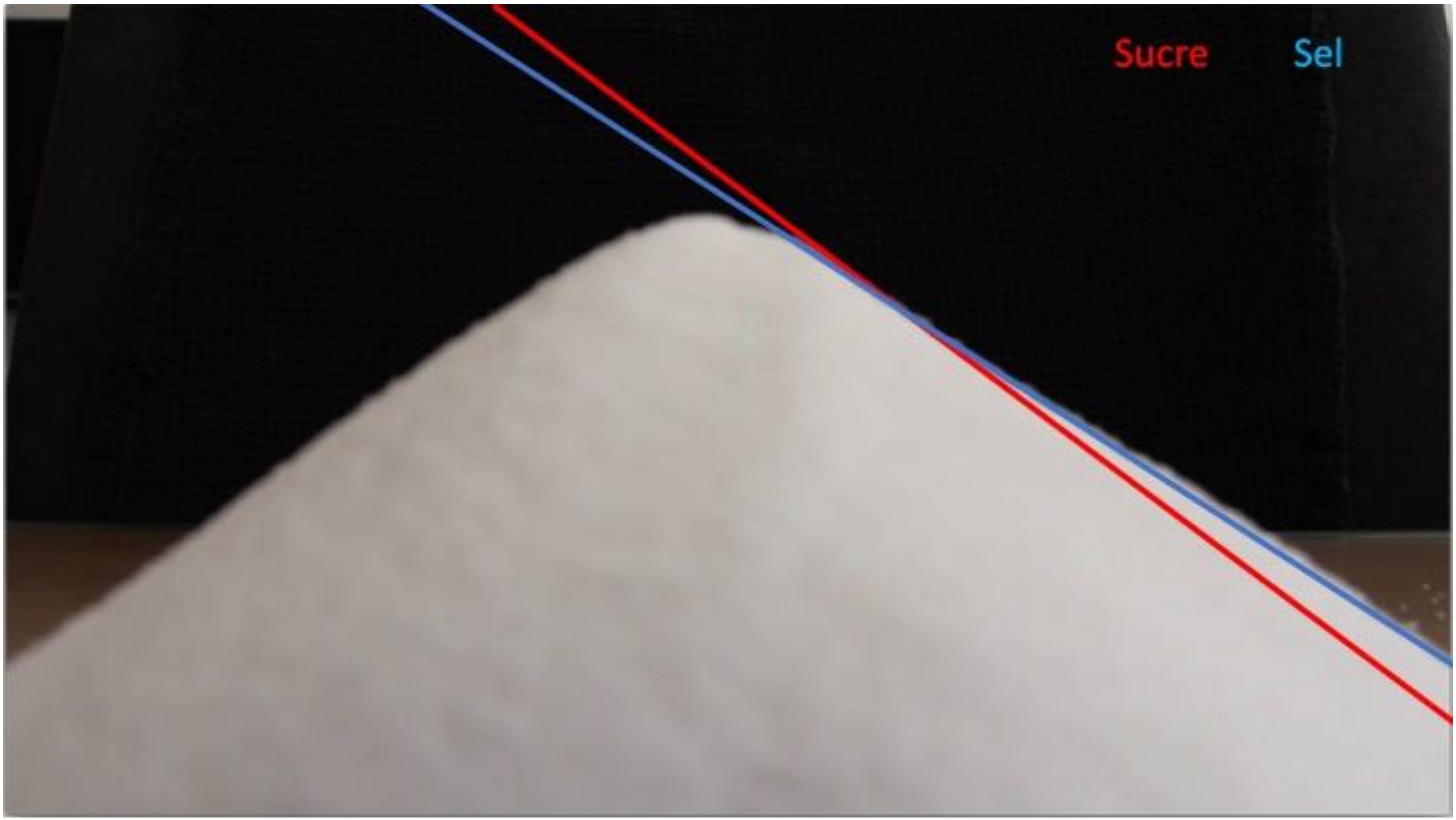

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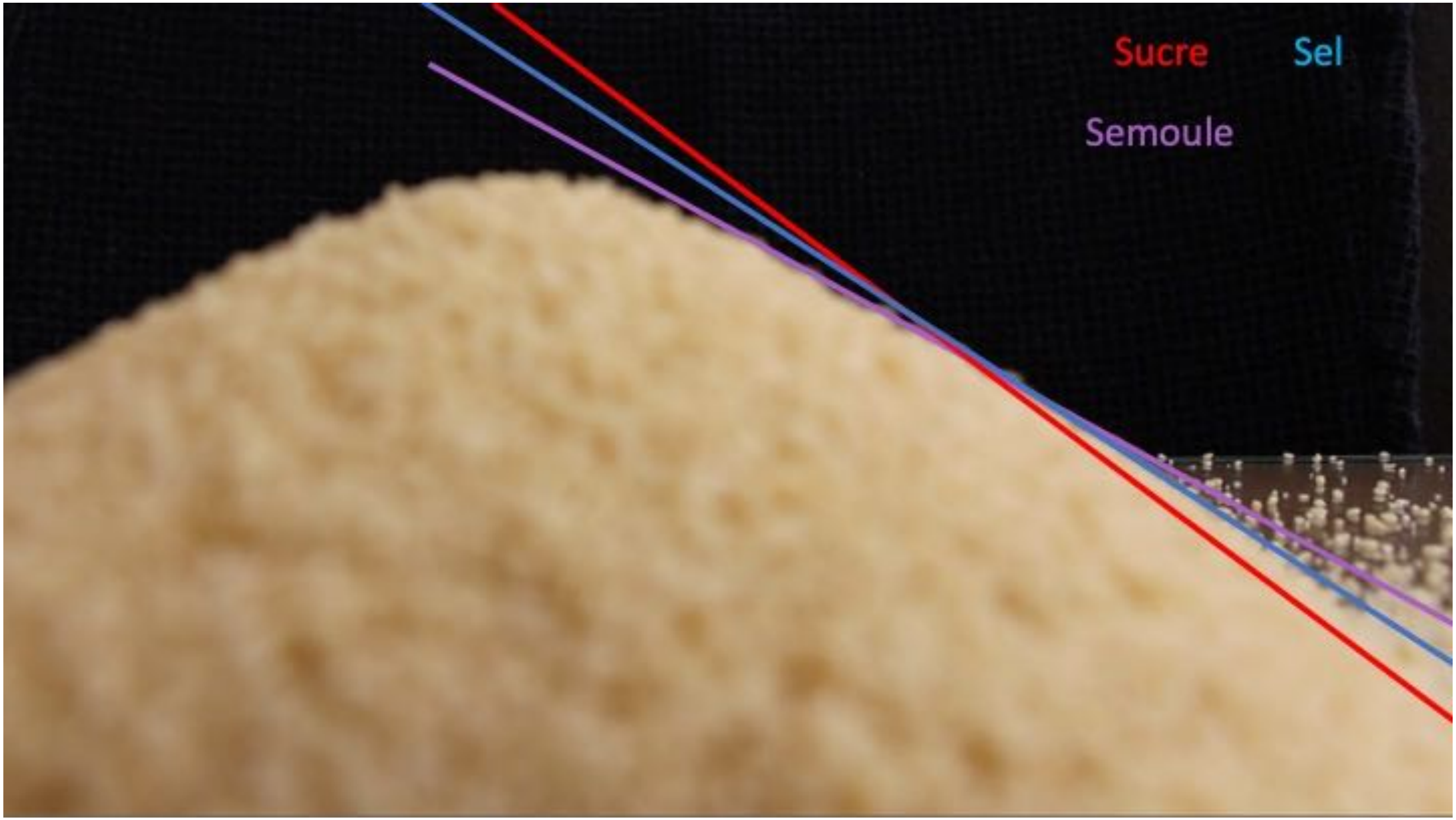

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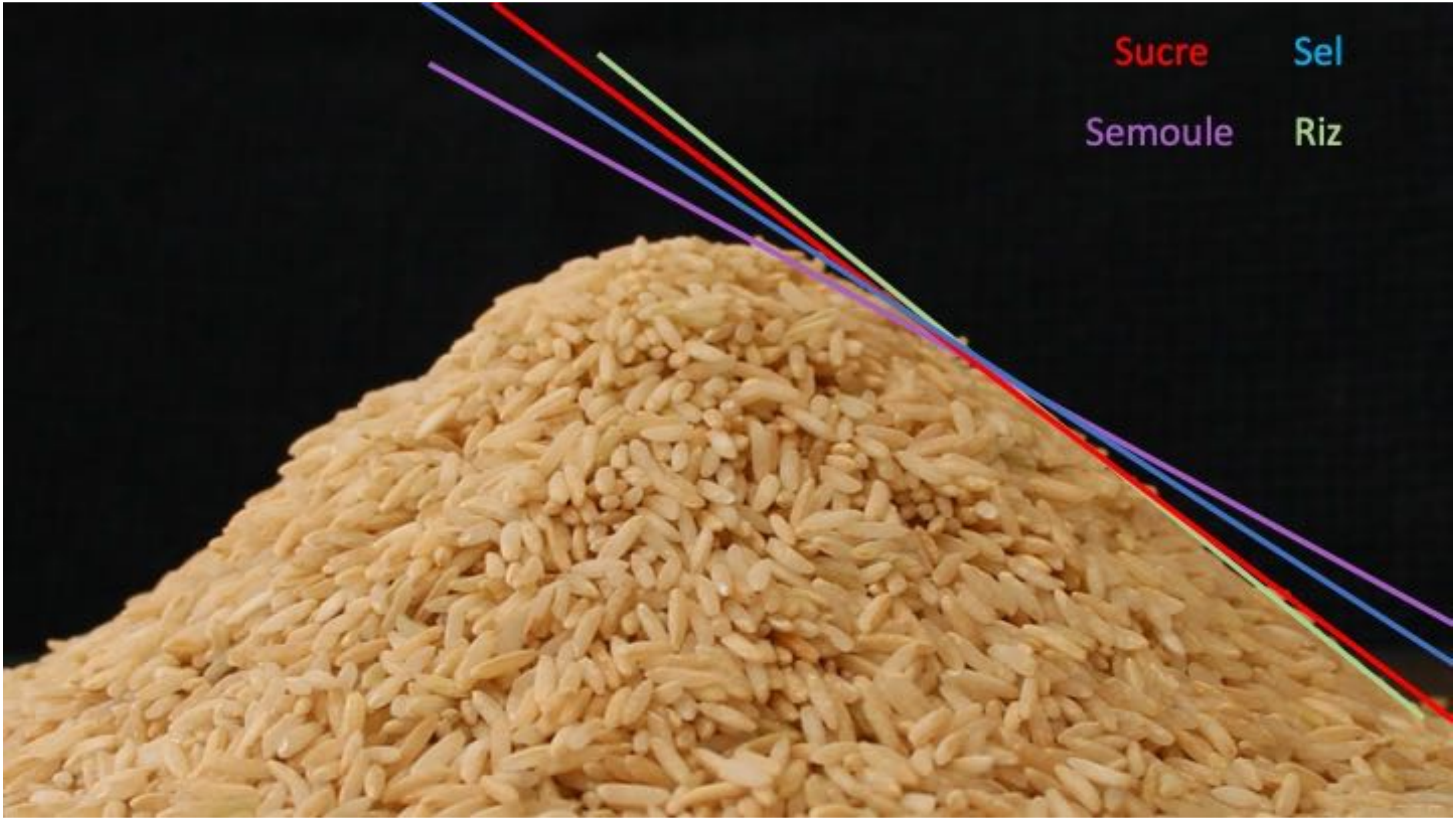


A complex behavior: solid, liquid or gaz?

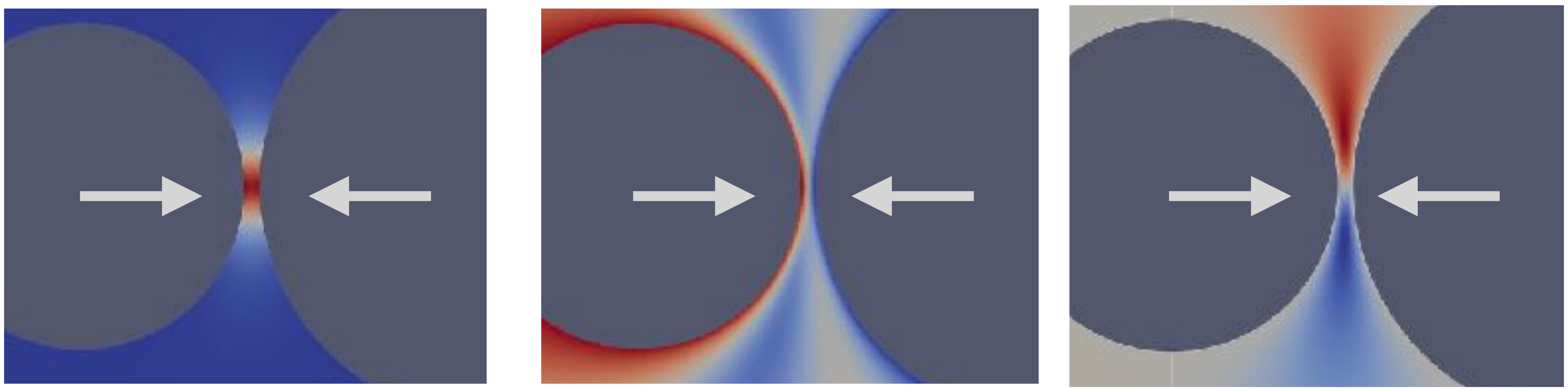
Sandcastle / Solid

Hourglass / Liquid

Sandstorm / Gaz

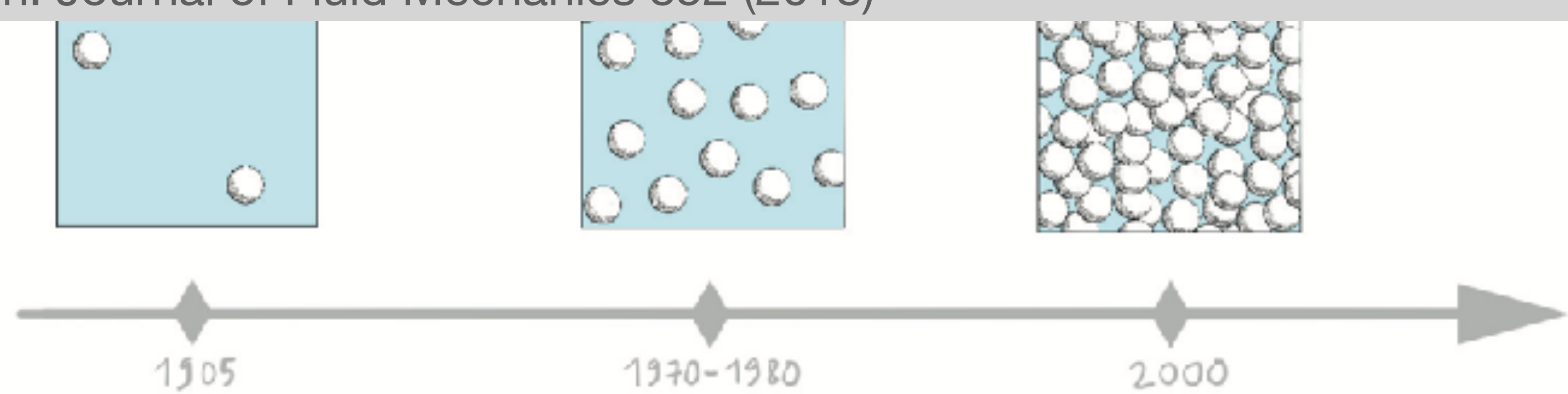
B. Andreotti, Y. Forterre, and O. Pouliquen. Granular media: between fluid and solid. Cambridge University Press, 2013

A complex behavior: lubrication in suspensions

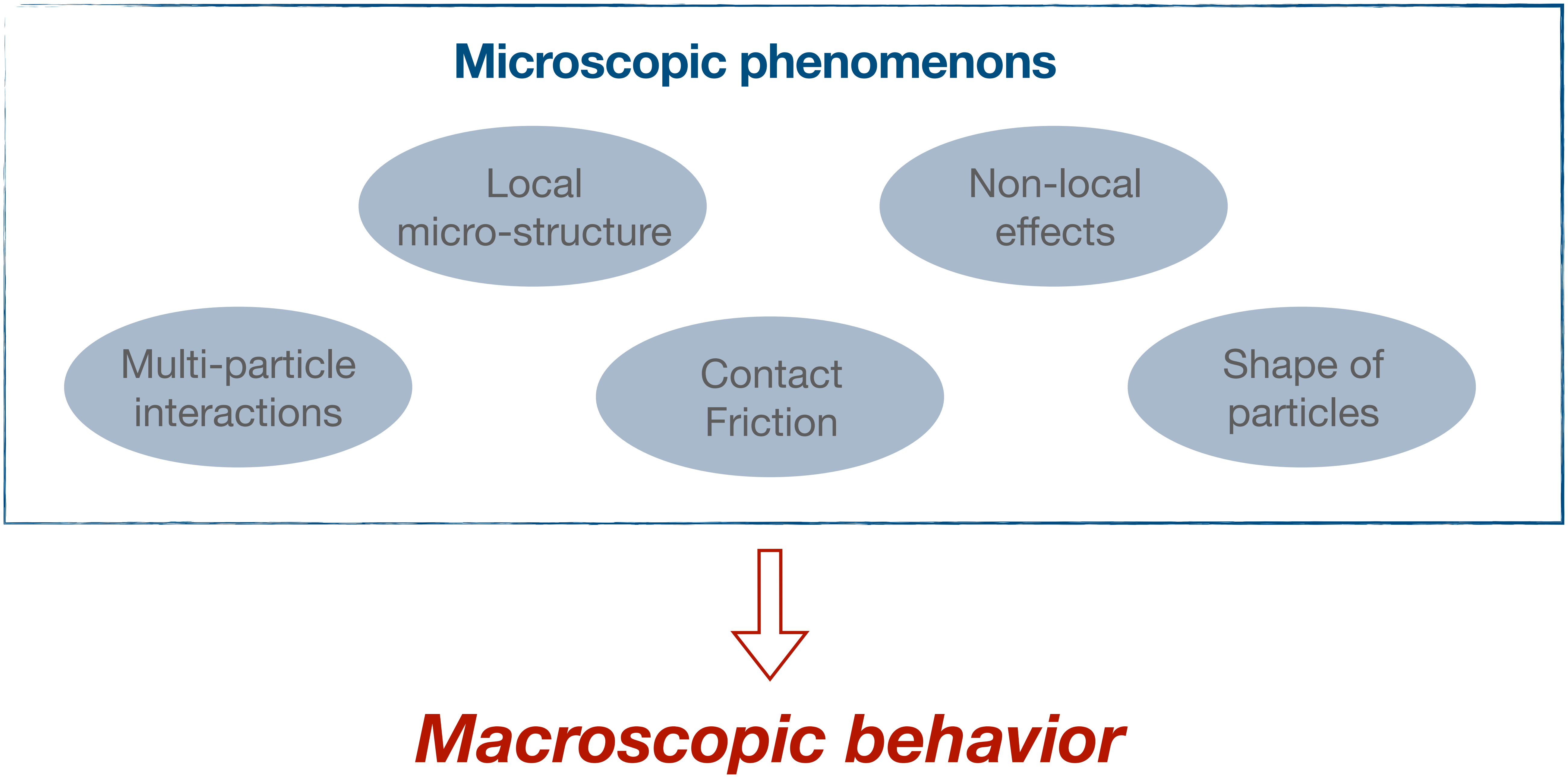


Pression — Vitesse horizontale — Vitesse verticale

E. Guazzelli and O. Pouliquen. “Rheology of dense granular suspensions”. In: Journal of Fluid Mechanics 852 (2018)



Methodology



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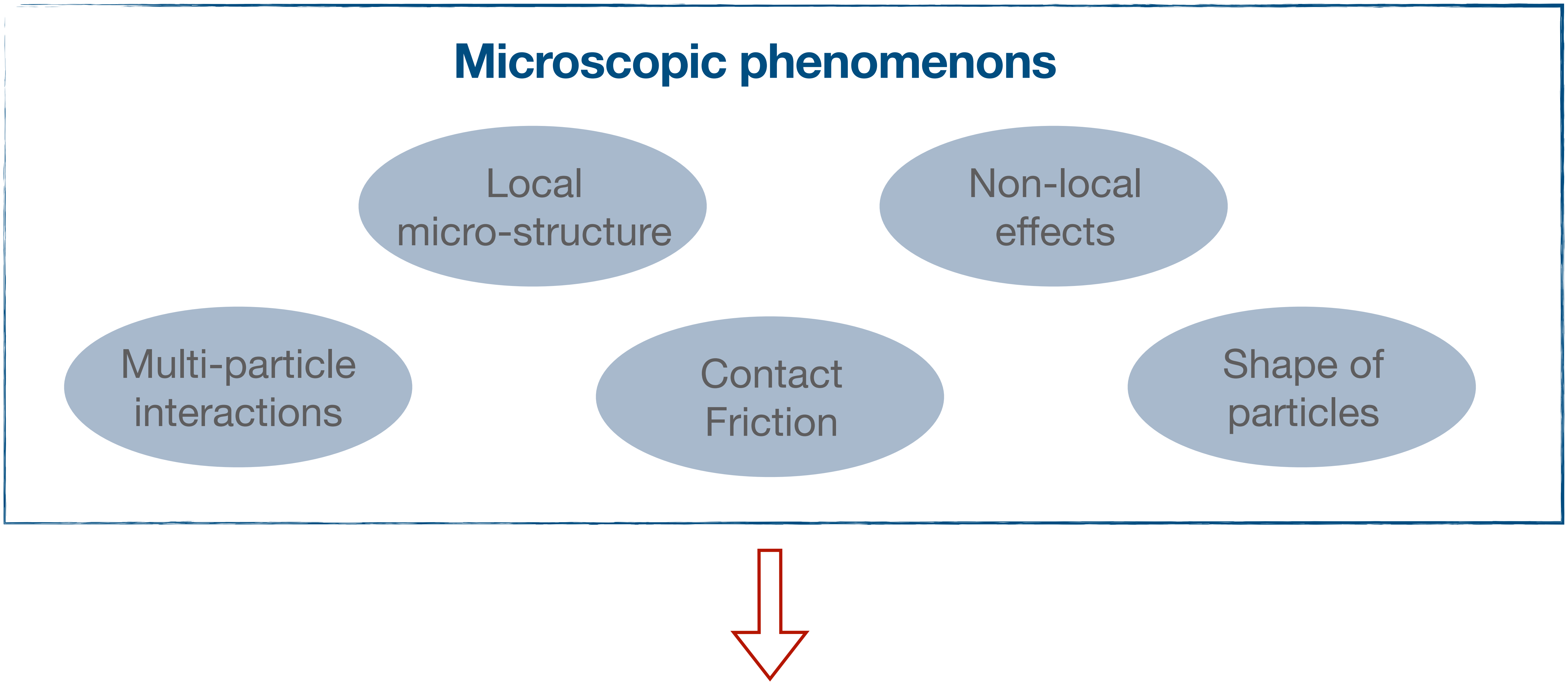


Macroscopic behavior

Numerical simulations at the microscopic level

Numerical simulations. The difficulties.

- Macroscopic behaviour

  $\Rightarrow$ Large number of particles

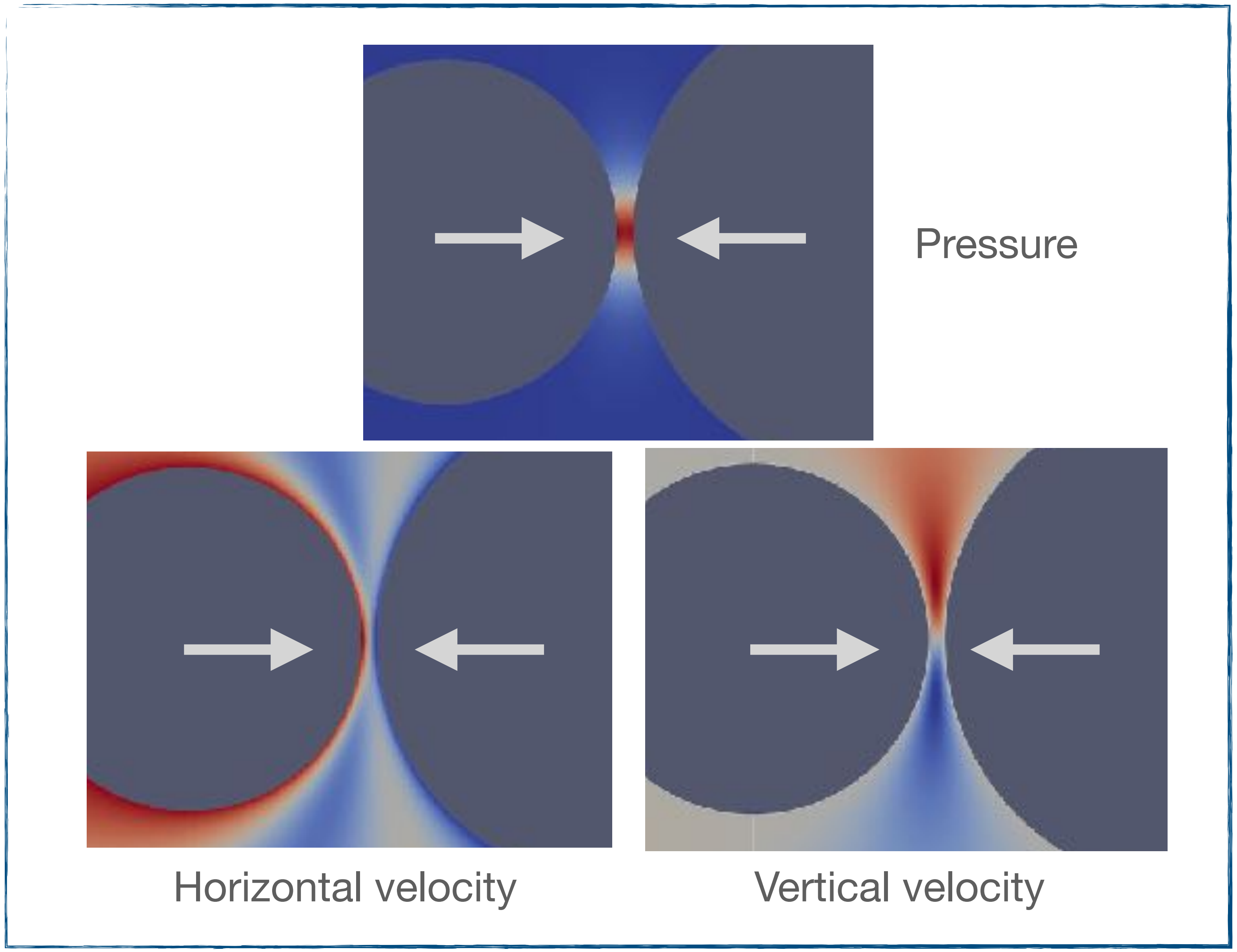
- Steady state and time average

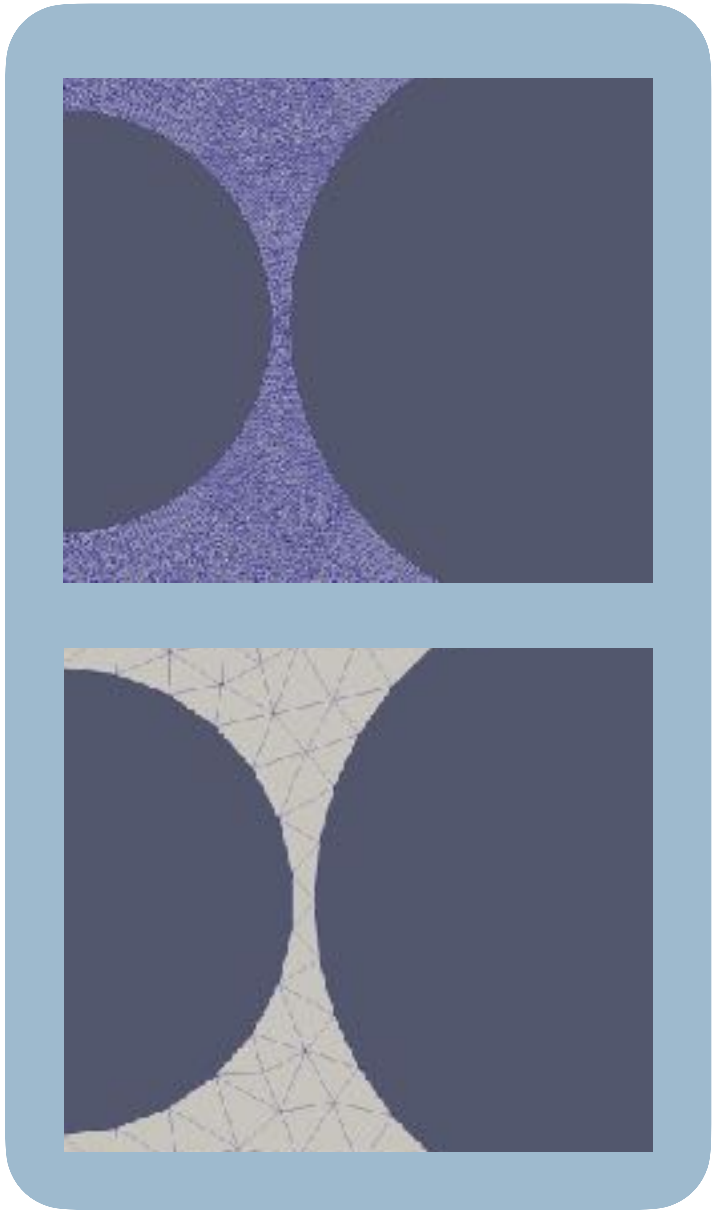
  $\Rightarrow$ Long time simulations

Need for fast N-body computations

Numerical simulations. The difficulties.

- Dense suspensions $\Rightarrow$ Close interactions due to the fluid



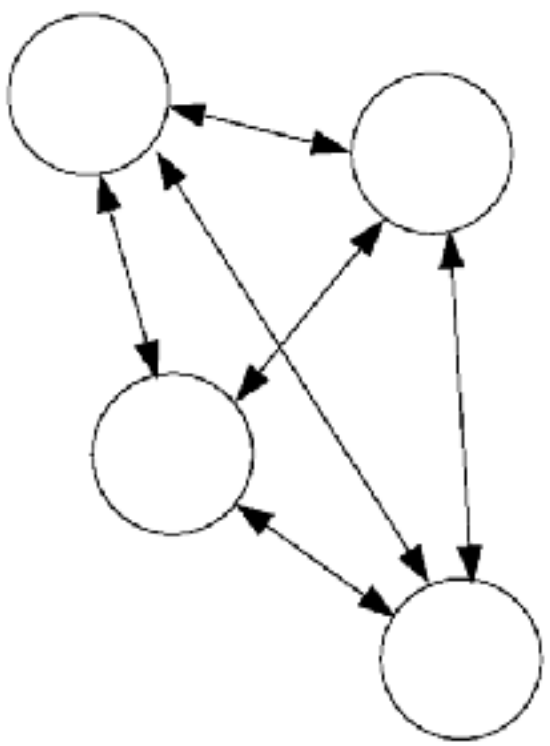


Need for methods taking lubrication into account

Numerical simulations. The difficulties.

- Dense suspensions
- Granular media

$\Rightarrow$ Solid contacts between particles



$$m_i \frac{d\dot{\mathbf{x}}_i}{dt} = F_i^{ext} + \sum_{i<j} F_{ij}^{contact}, \quad i = 1 \ldots N$$

- Multi-particle interactions
- Stiff interactions / non-continuous forces, depending on the model.

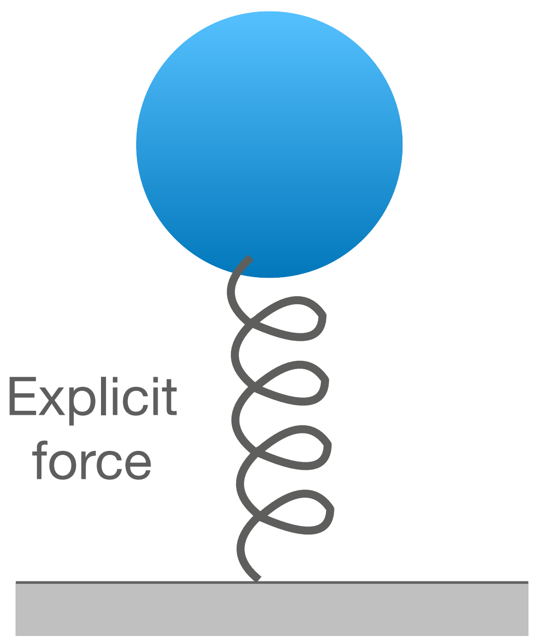
Need for a stable algorithm to deal with contacts

Numerical simulations. The difficulties.

- Dense suspensions
- Granular media

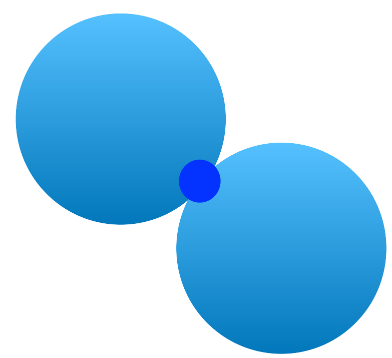
$\Rightarrow$ Solid contacts between particles

Explicit penalty force



Continuous penalization method

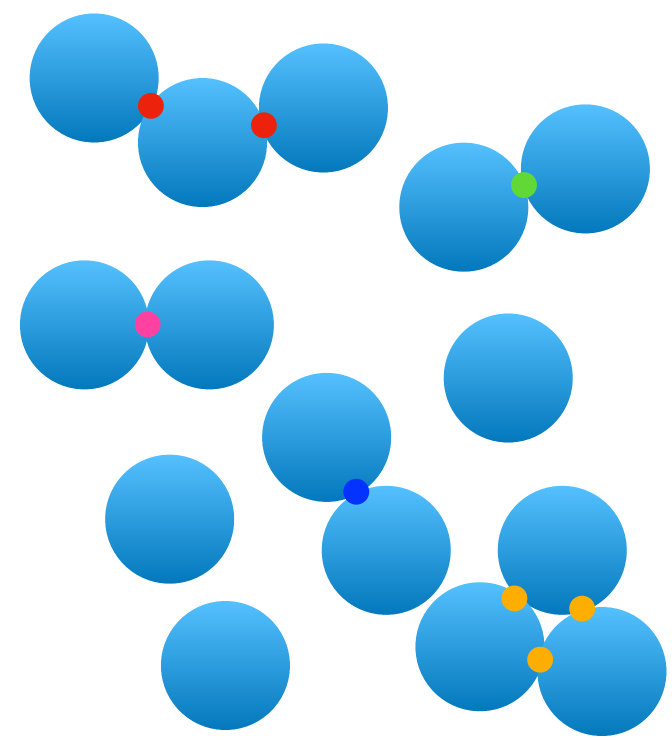
Contact law



Inelastic contacts
Coulomb friction law

Explicit solution for 2 particles

Multi-contact problem

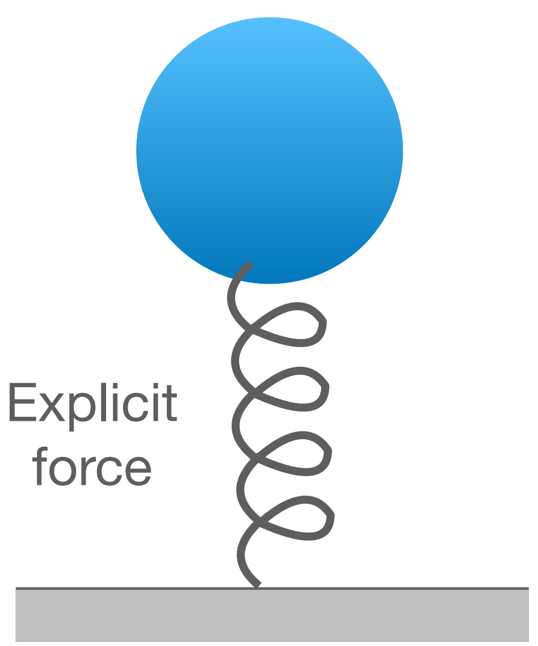


Numerical simulations. The difficulties.

- Dense suspensions
- Granular media

⇒ Solid contacts between particles

Explicit penalty force

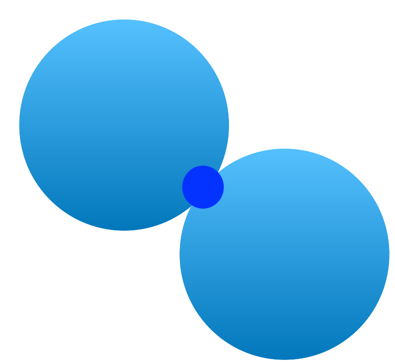


Continuous penalization method

Molecular Dynamics

[Cundall, Strack, 1979]

Contact law



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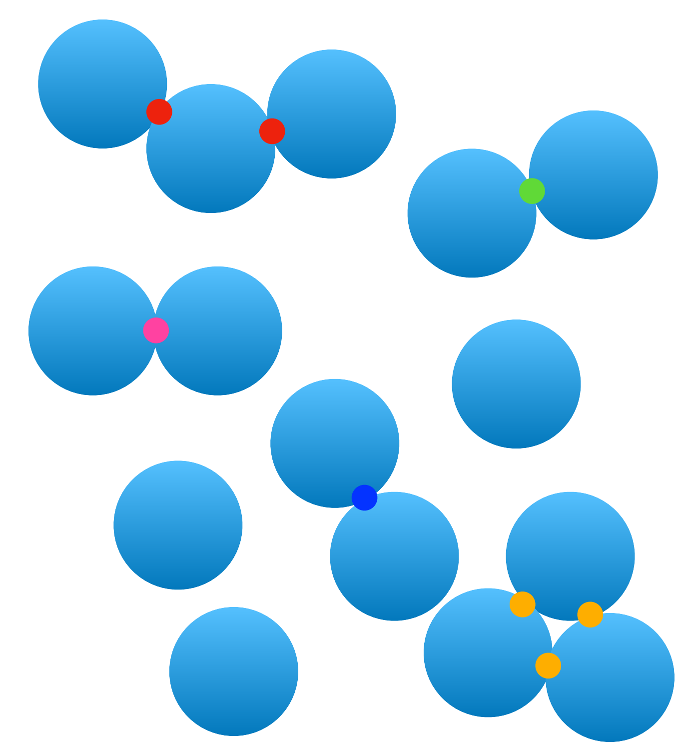

Numerical simulations. The difficulties.

- Dense suspensions
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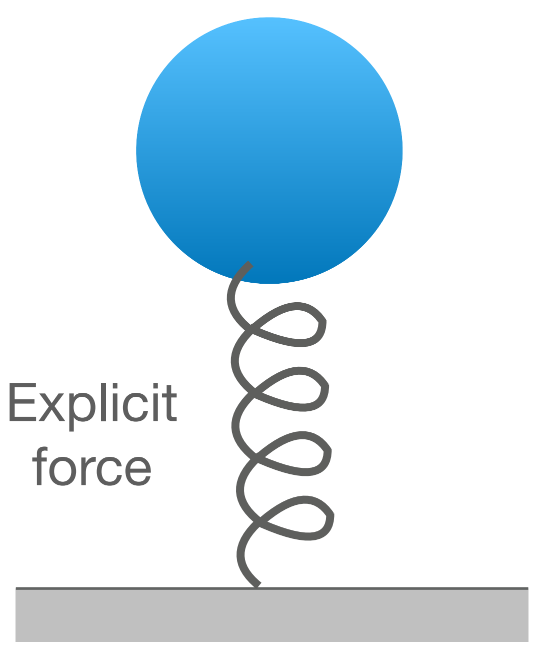
$\Rightarrow$ Solid contacts between particles

Explicit penalty force



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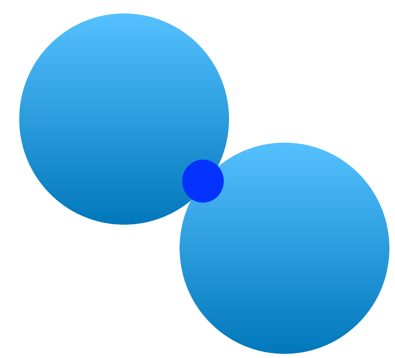
Contact law



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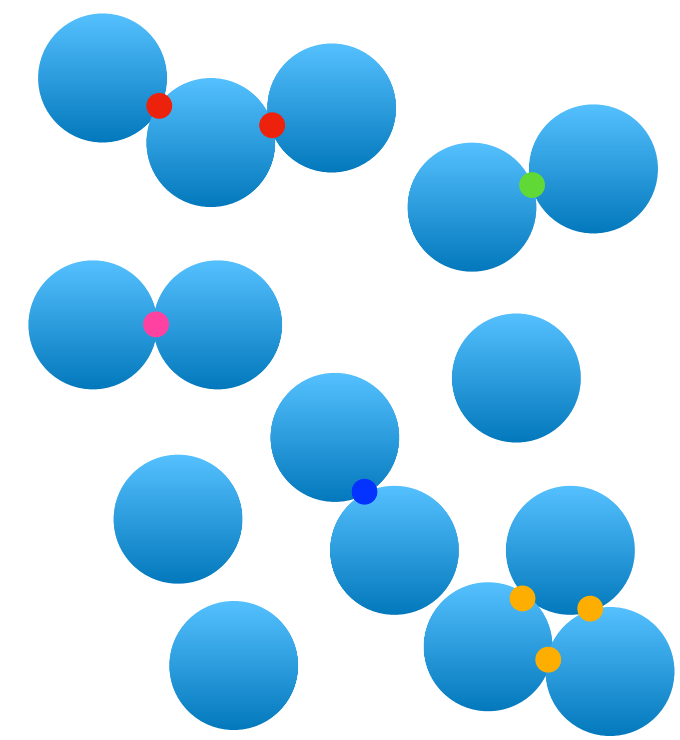


Molecular Dynamics

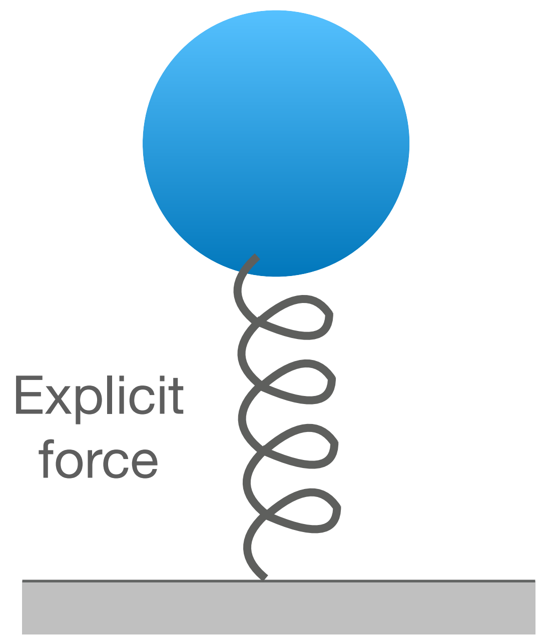
[Cundall, Strack, 1979]

Non-Smooth Contact Dynamics

[Moreau, 1988] [Moreau, Jean, 1992]

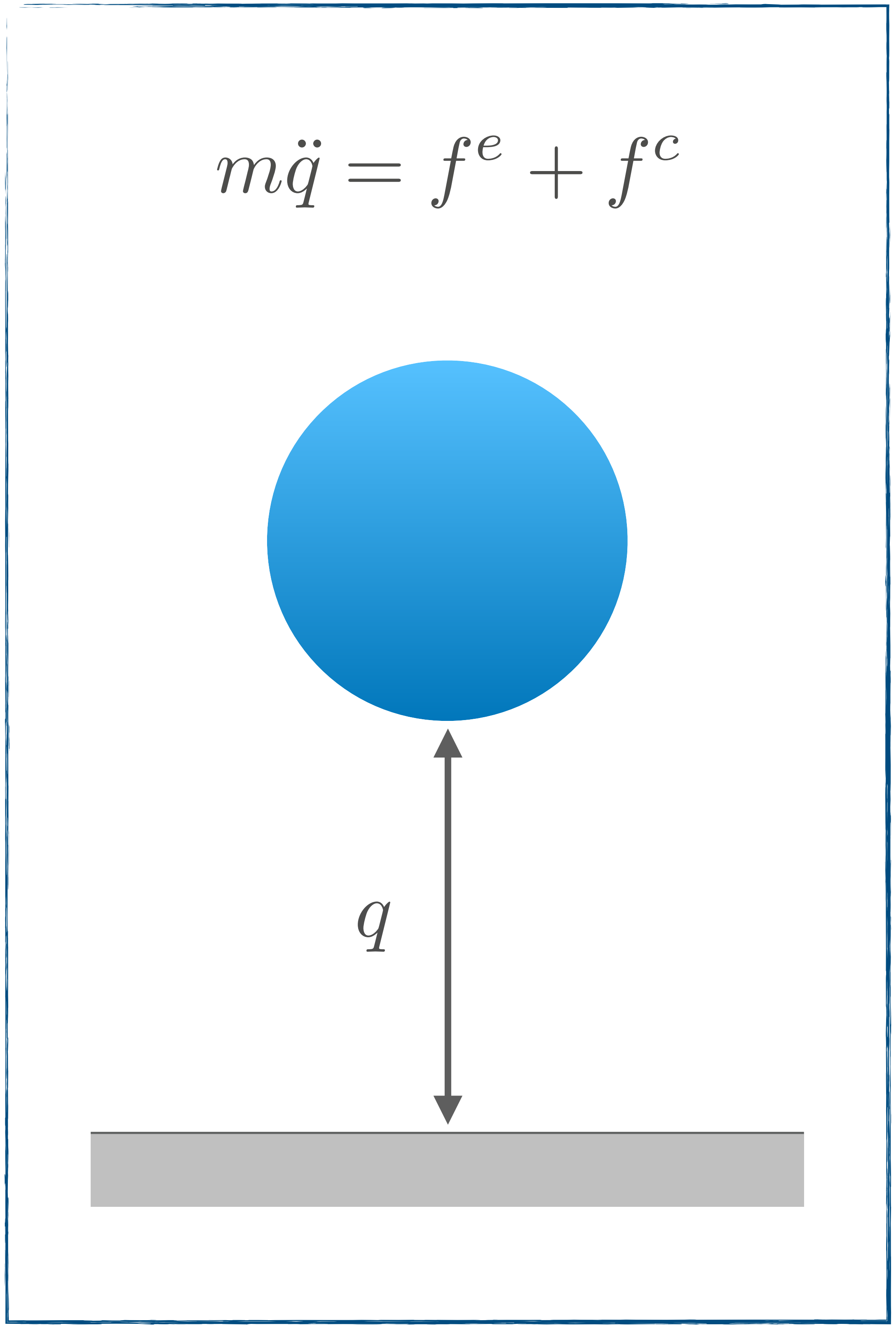
Frictionless Molecular Dynamics Models

Explicit penalty force



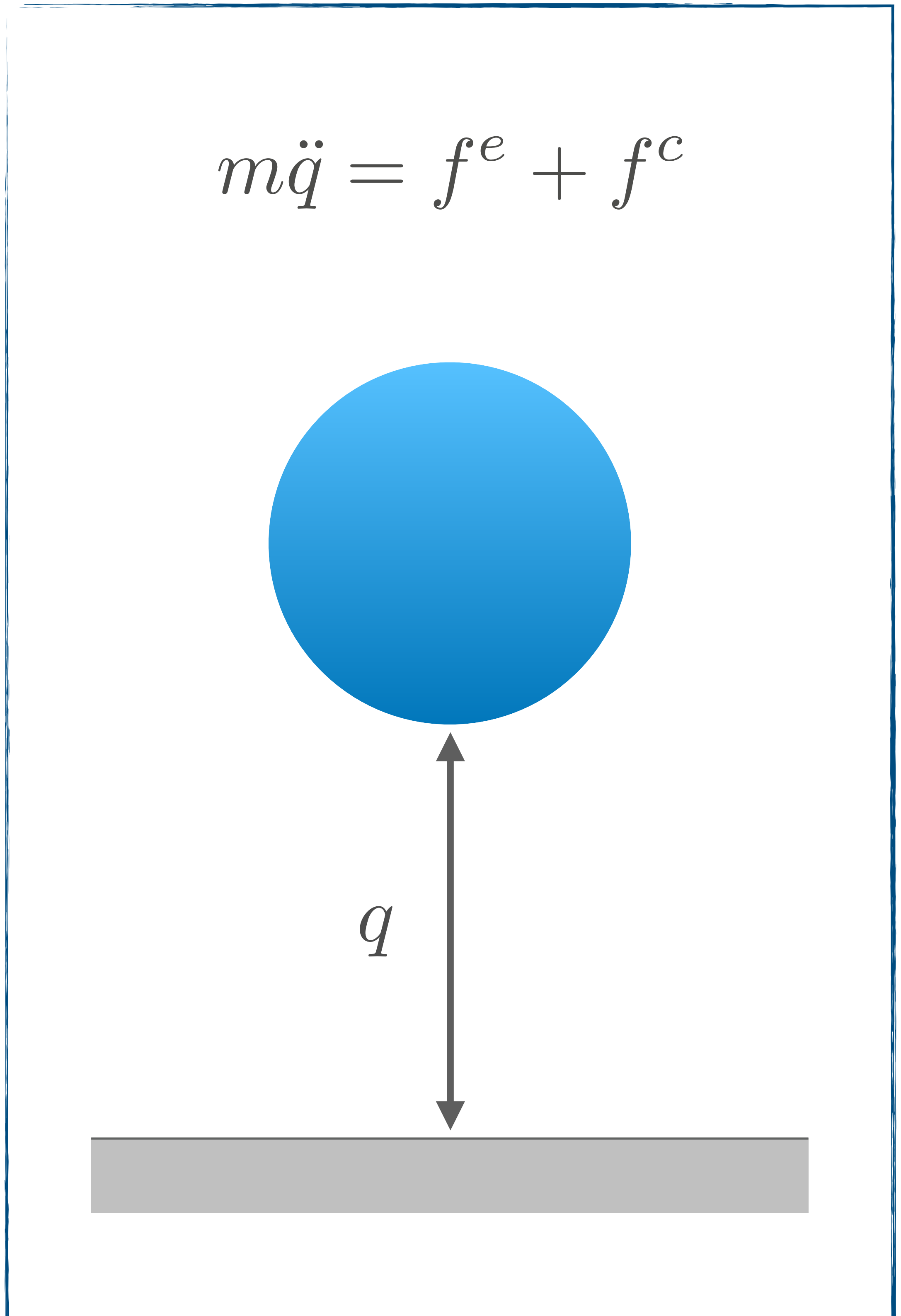
Continuous penalization method

An explicit contact force



- Contact forces explicit
- Which choice for the contact model?

An explicit contact force



- Linear spring model

$$f^c(q) = 0 \qquad \text{if } q \geq 0$$
$$= -k\,q \qquad \text{if } q \leq 0$$

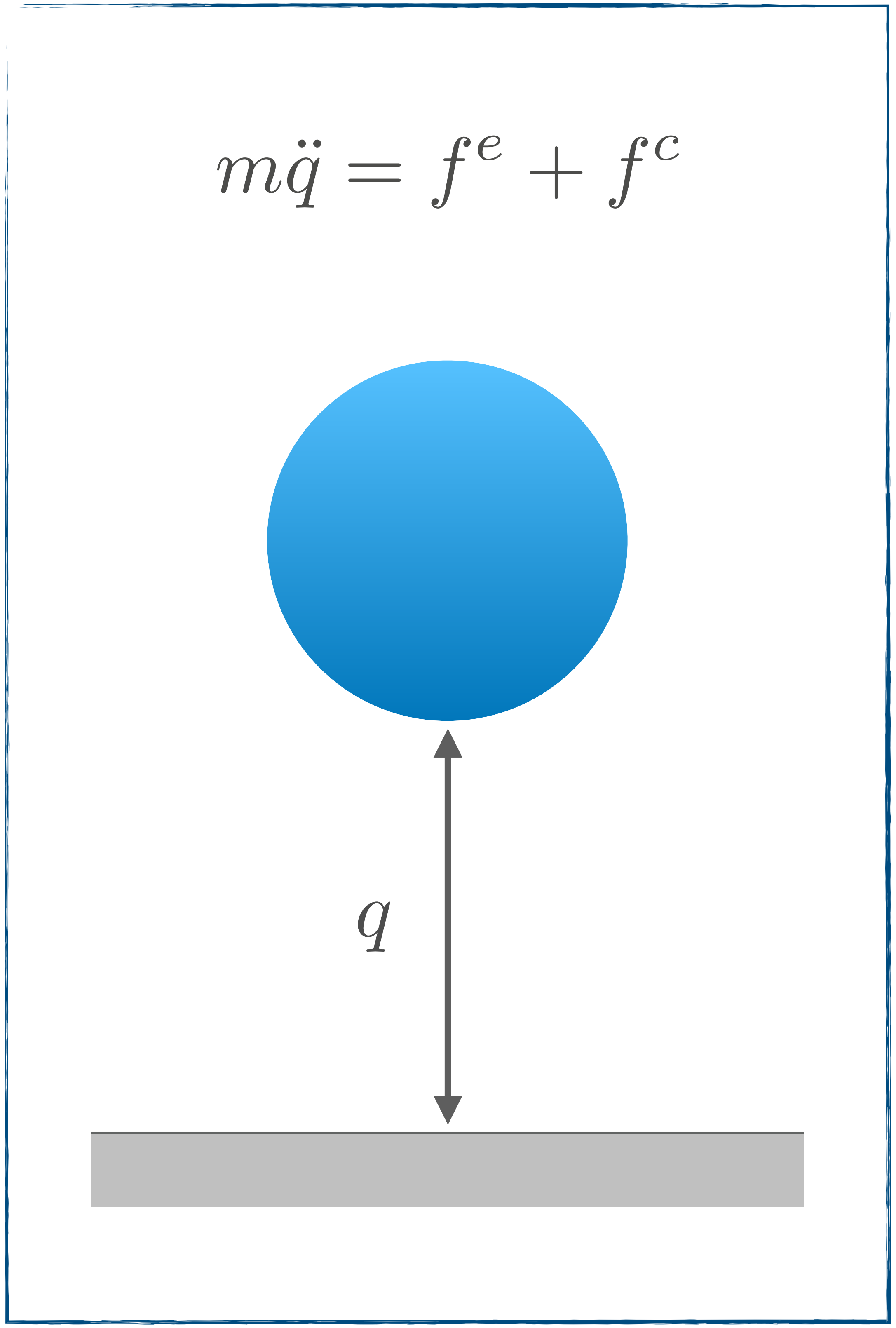
When $k \approx$ grain rigidity $\longrightarrow +\infty$

Signorini conditions.

$$q \geq 0, \quad f^c(q) \geq 0$$
$$q\, f^c(q) = 0$$

⚠ Negative contact force…
Missing energy dissipation…

An explicit contact force



Damped linear spring model

$$f^c(q) = -k\,q - \gamma\,\frac{dq}{dt}$$

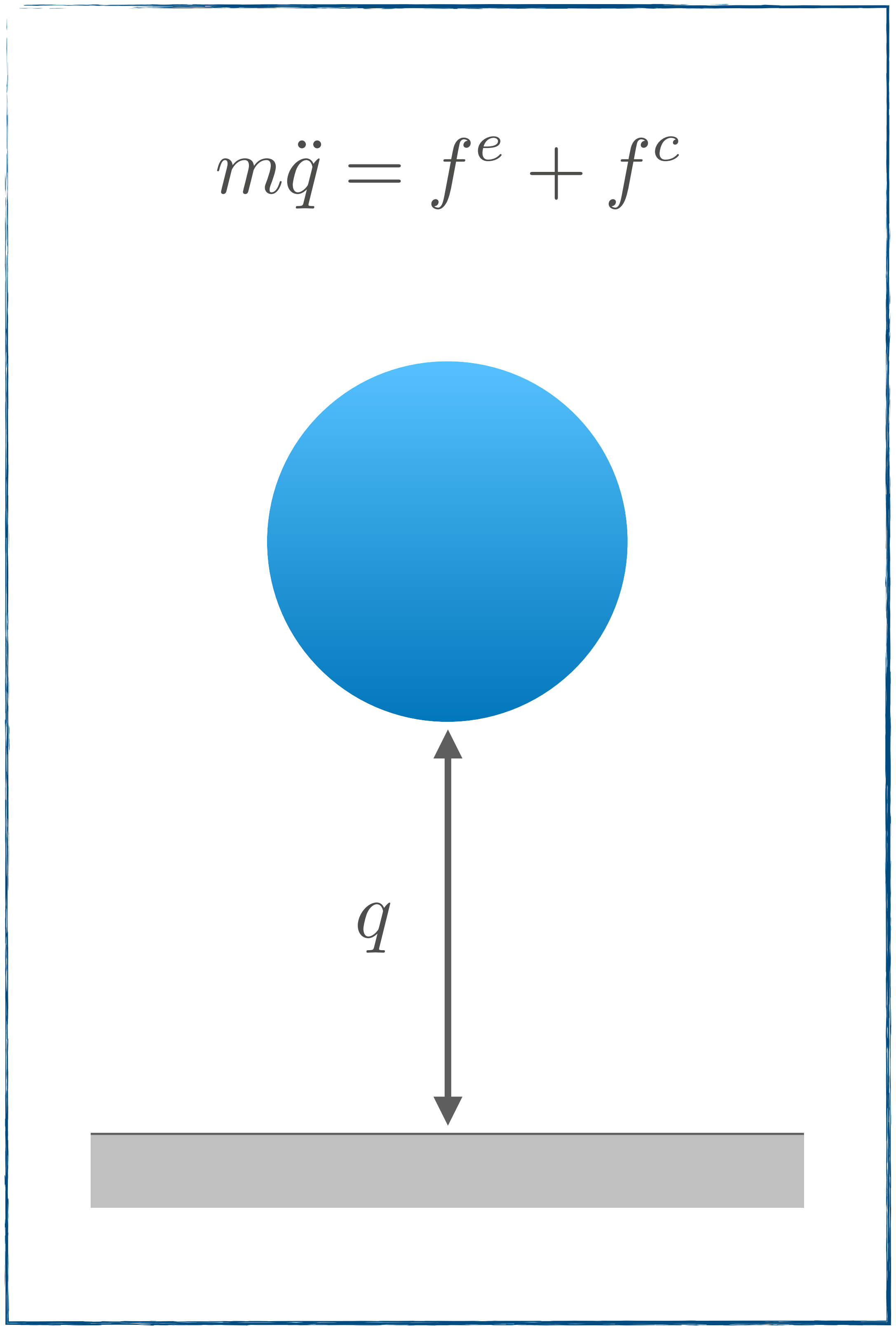
Damped harmonic oscillator:

$$\frac{d^2q}{dt^2} = -\omega_0^2(q - q_0) - 2\mu_0\,\frac{dq}{dt}$$

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \mu_0 = \frac{\gamma}{2m}, \quad q_0 = -\frac{mg}{k}$$

Negative contact force…
Fine tuning of parameters

An explicit contact force



- Positive force

$$\begin{aligned} f^c(q) &= 0 && \text{if } q \geq 0 \\ &= \max(0, \cdot) && \text{if } q \leq 0 \end{aligned}$$

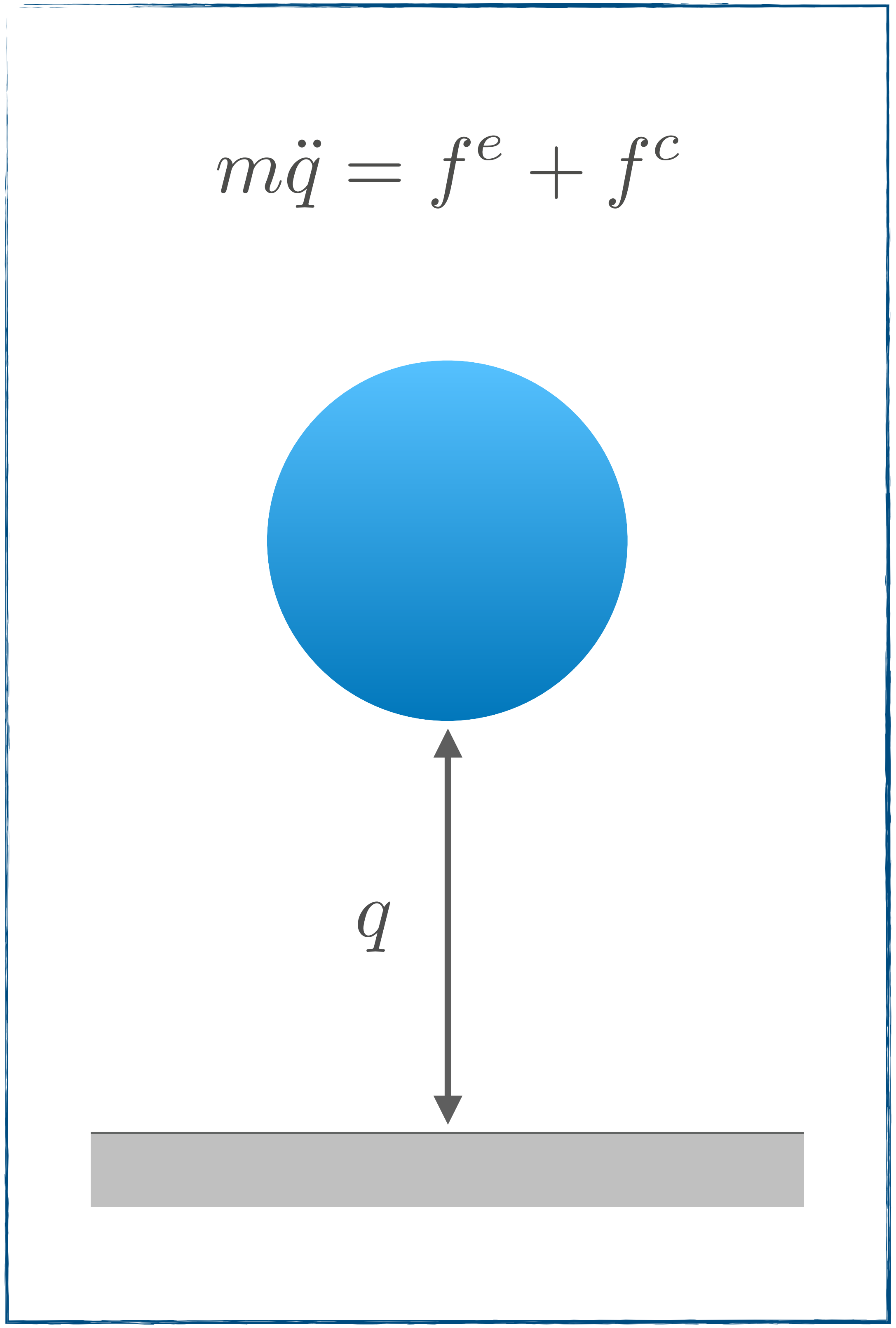
- Many non-linear contact laws
  Ex: Hertz contact law

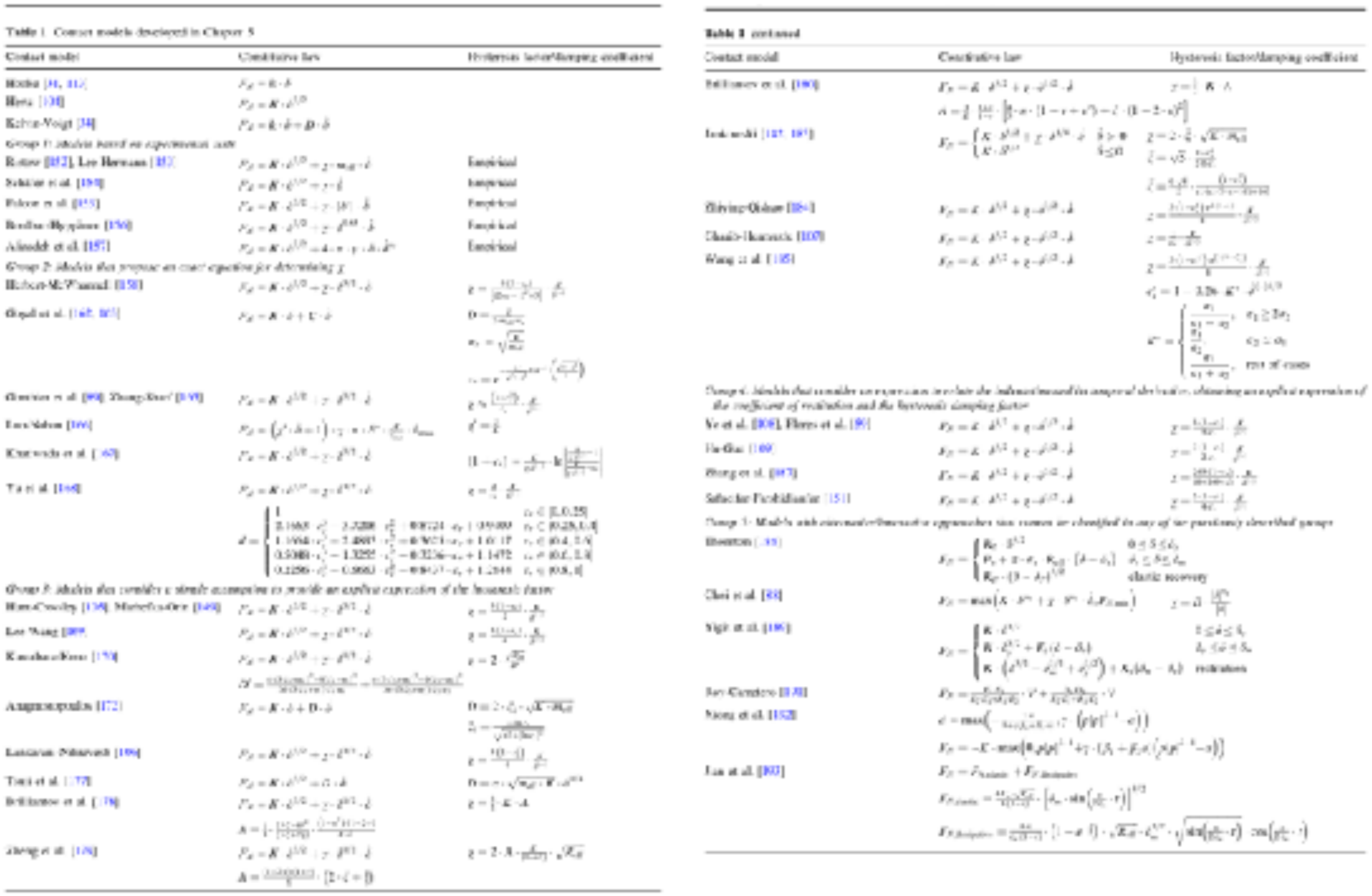
$$f^c(q) = \frac{E\sqrt{2r}}{3(1-\nu^2)} q^{3/2}$$

- Repulsive force, adhesion force, variable parameters …

- Tangential force

An explicit contact force





[E. Corral, R.G. Moreno, M.J.G. Garcıa, and C. Castejon. Nonlinear phenomena of contact in multibody systems dynamics: a review. Nonlinear Dynamics, 2021.]

Further extensions

- Multi-particle case
  - Obvious extension of the model
  - Need for neighbour search algorithms
  - Need for additional dissipation to dampen long waves

- Non-spherical particles
  - Composed of assemblies of spheres
  - Polygons (various contact scenarios)

Time discretization

- Stiff second order ODE to solve

- The forces are not time-derivable
  - No high-order scheme

- Mainly used numerical schemes:
  - Leap-frog scheme
  - Predictor/corrector schemes

Summary

- Stiff explicit second order ODE model

Pros

- Easy to implement
- Many contact models available

Cons

- Fine tuning of parameters
- Small time steps
- Interpreting the results can be difficult

Bibliography

- P.A. Cundall and O.D.L. Strack. A discrete numerical model for granular assemblies. Geotechnique, 29(1):47–65, 1979. Publisher: ICE Publishing.

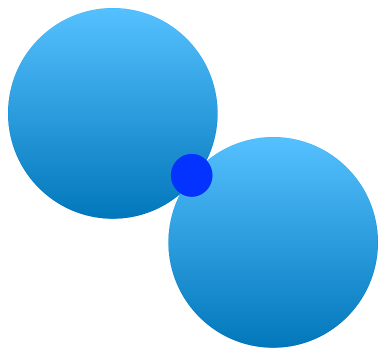
- S. Luding. Introduction to discrete element methods. European Journal of Environmental and Civil Engineering, 12(7-8):785–826, 2008

- E. Corral, R.G. Moreno, M.J.G. Garcıa, and C. Castejon. Nonlinear phenomena of contact in multibody systems dynamics: a review. Nonlinear Dynamics, 2021.

- F. Radjai, F. Dubois (2011), Discrete-element modeling of granular materials, (Wiley-Iste, Berlin), 425 pages, ISBN 978-1-84821-260-2

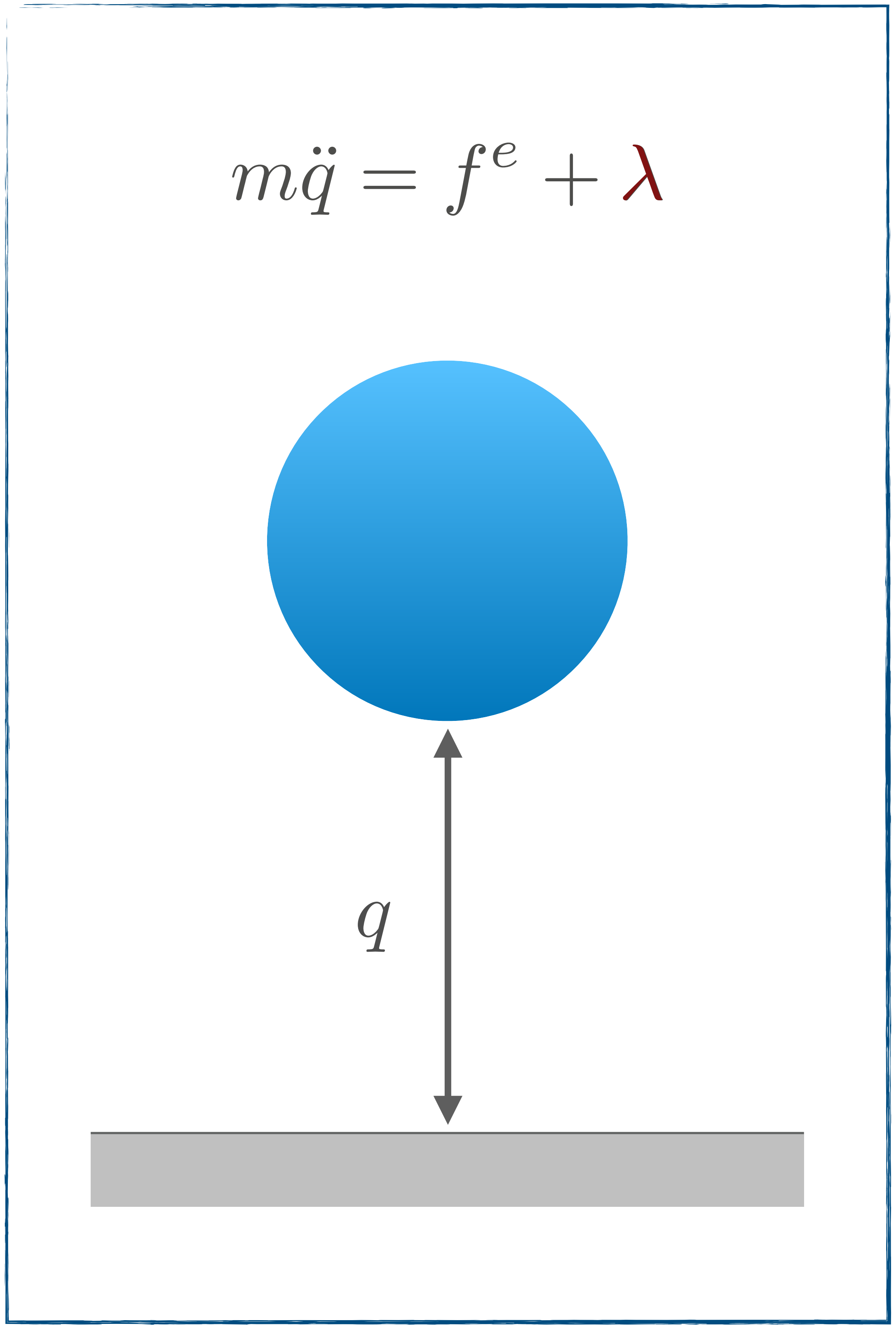
Frictionless Contact Dynamics Models

Contact law



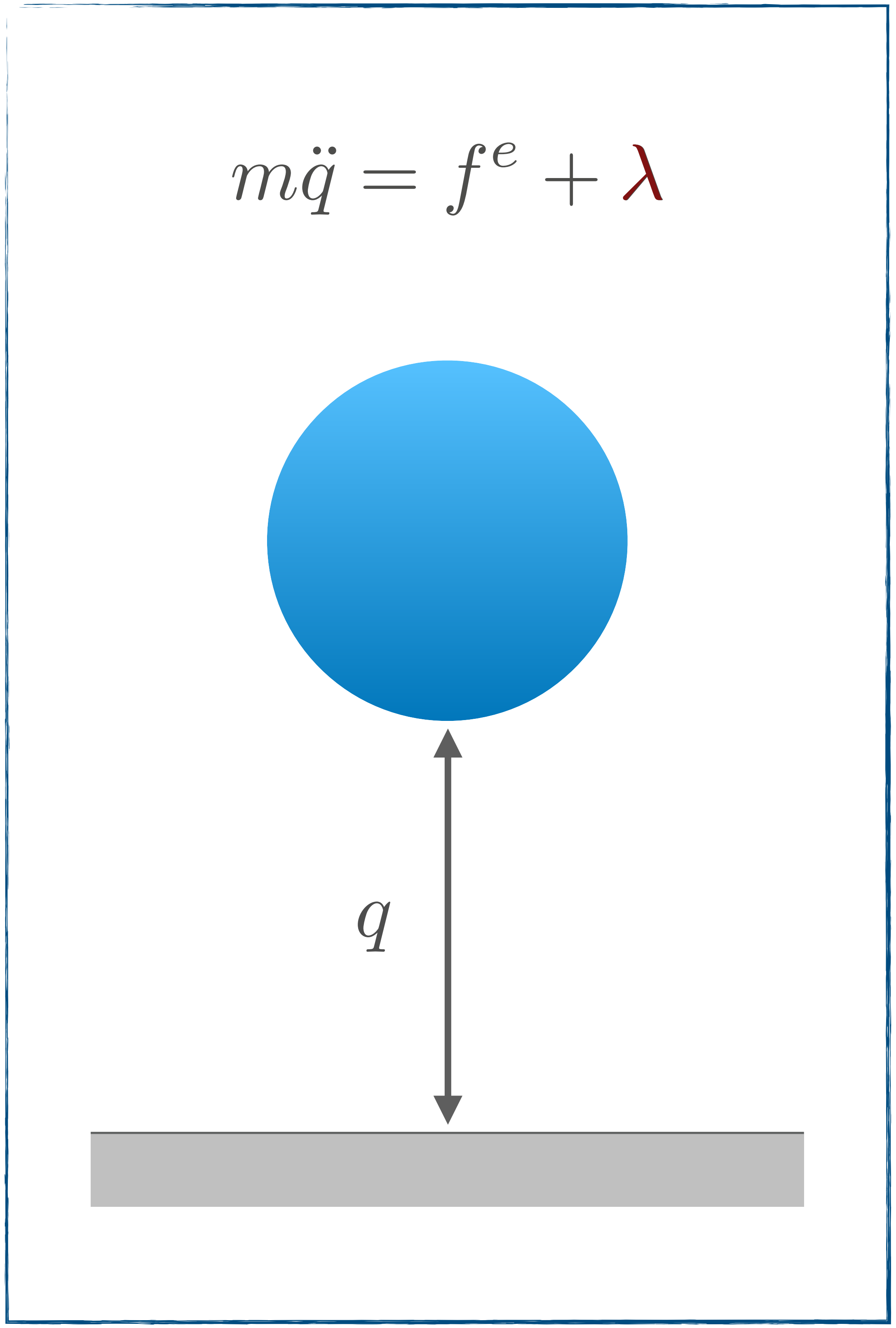
Inelastic contacts

An implicit contact force, a constrained problem



- Contact forces implicit
- How to determine λ ?

An implicit contact force, a constrained problem



$$m\ddot{q} = f^e + \lambda$$

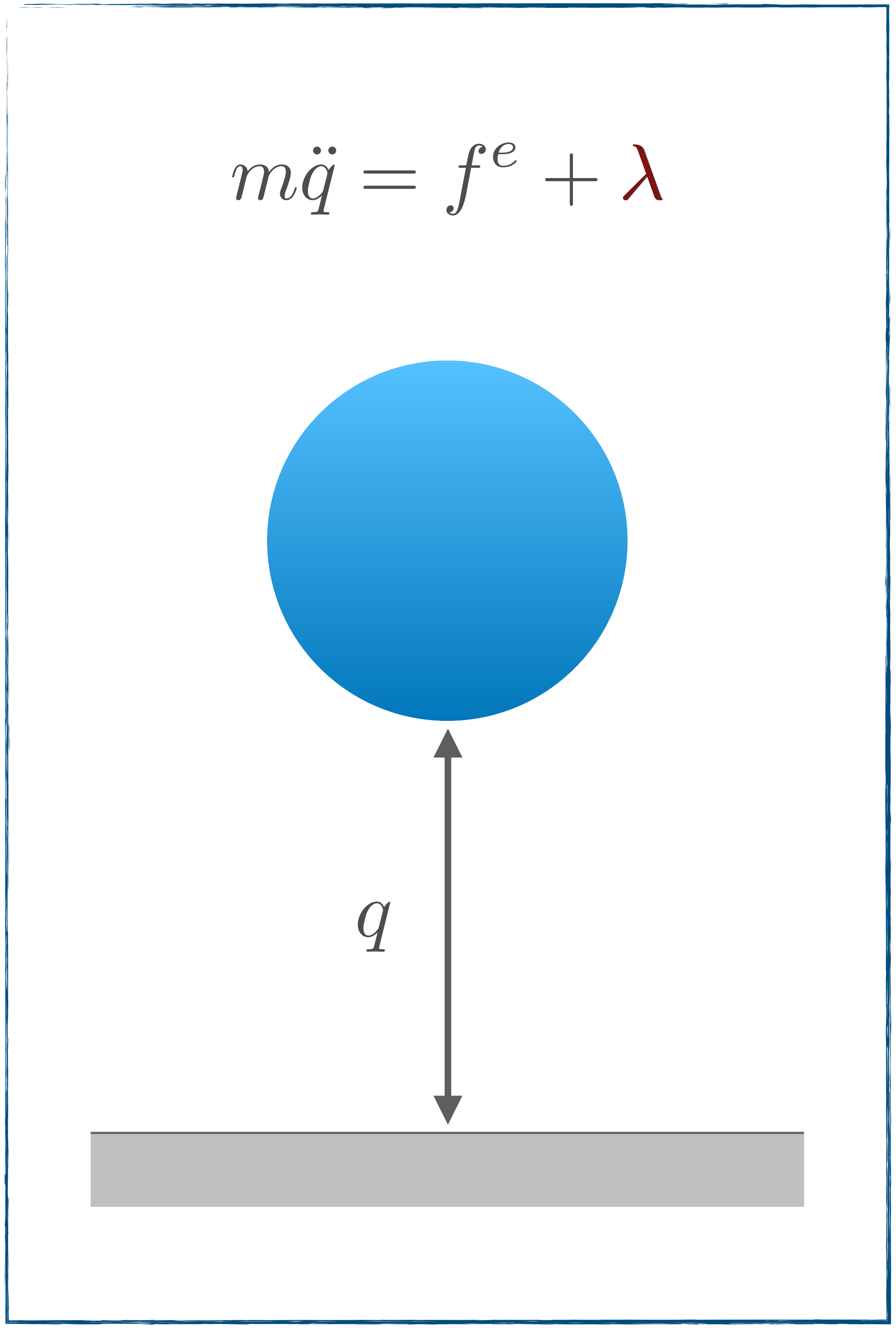
$$\mathrm{supp}(\lambda) \subset \{t \,/\, q(t) = 0\}$$

$$q \geq 0, \quad \lambda \geq 0$$

$$\dot{q}^+ = P_{C_q}(\dot{q}^-)$$

$$C_q = \{v \in \mathbb{R} \,/\, v \geq 0 \ \text{ if } \ q = 0\}$$

An implicit contact force, a constrained problem



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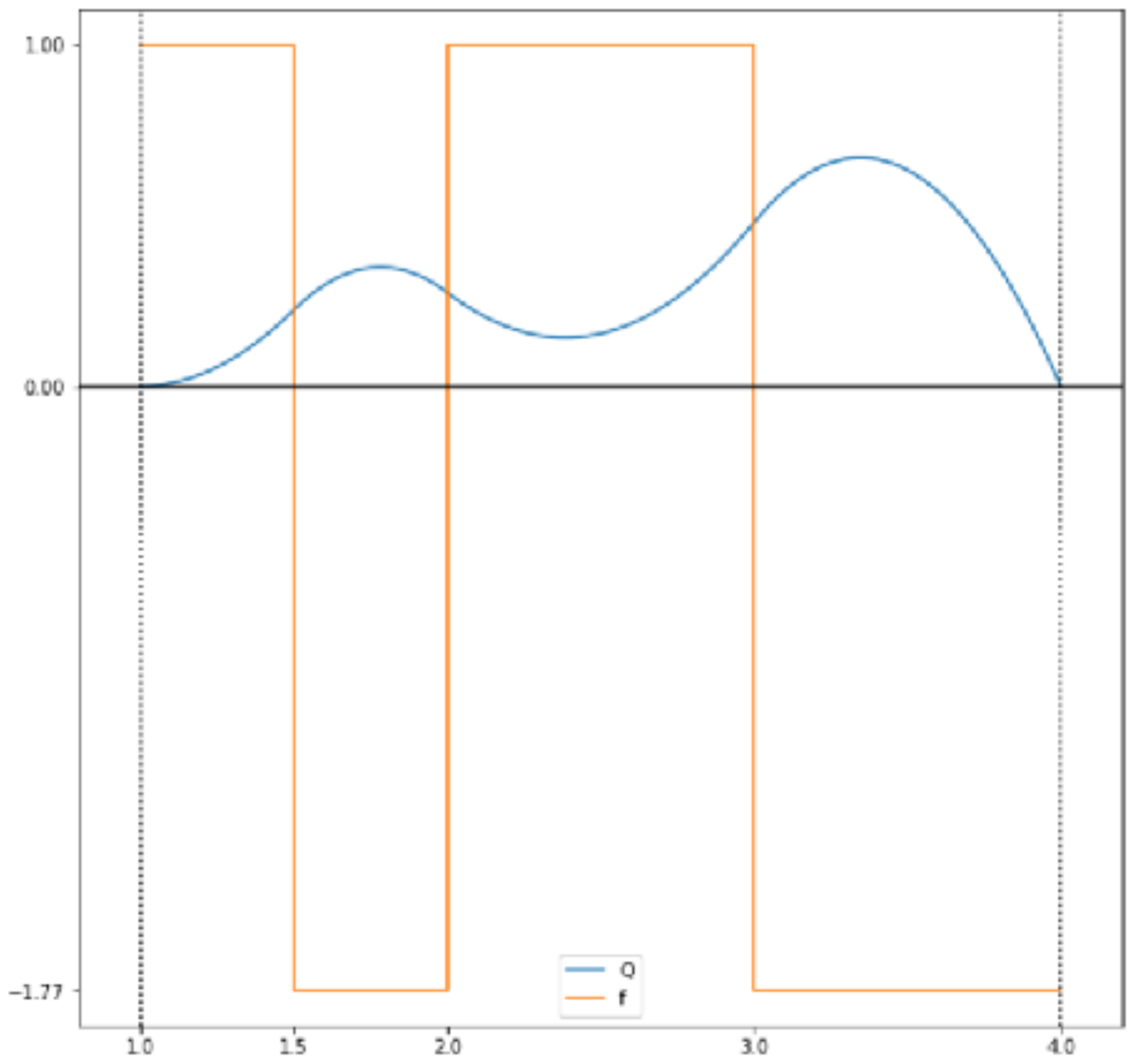
$$C_q = \{v \in \mathbb{R} \,/\, v \geq 0 \ \text{ if } \ q = 0\}$$

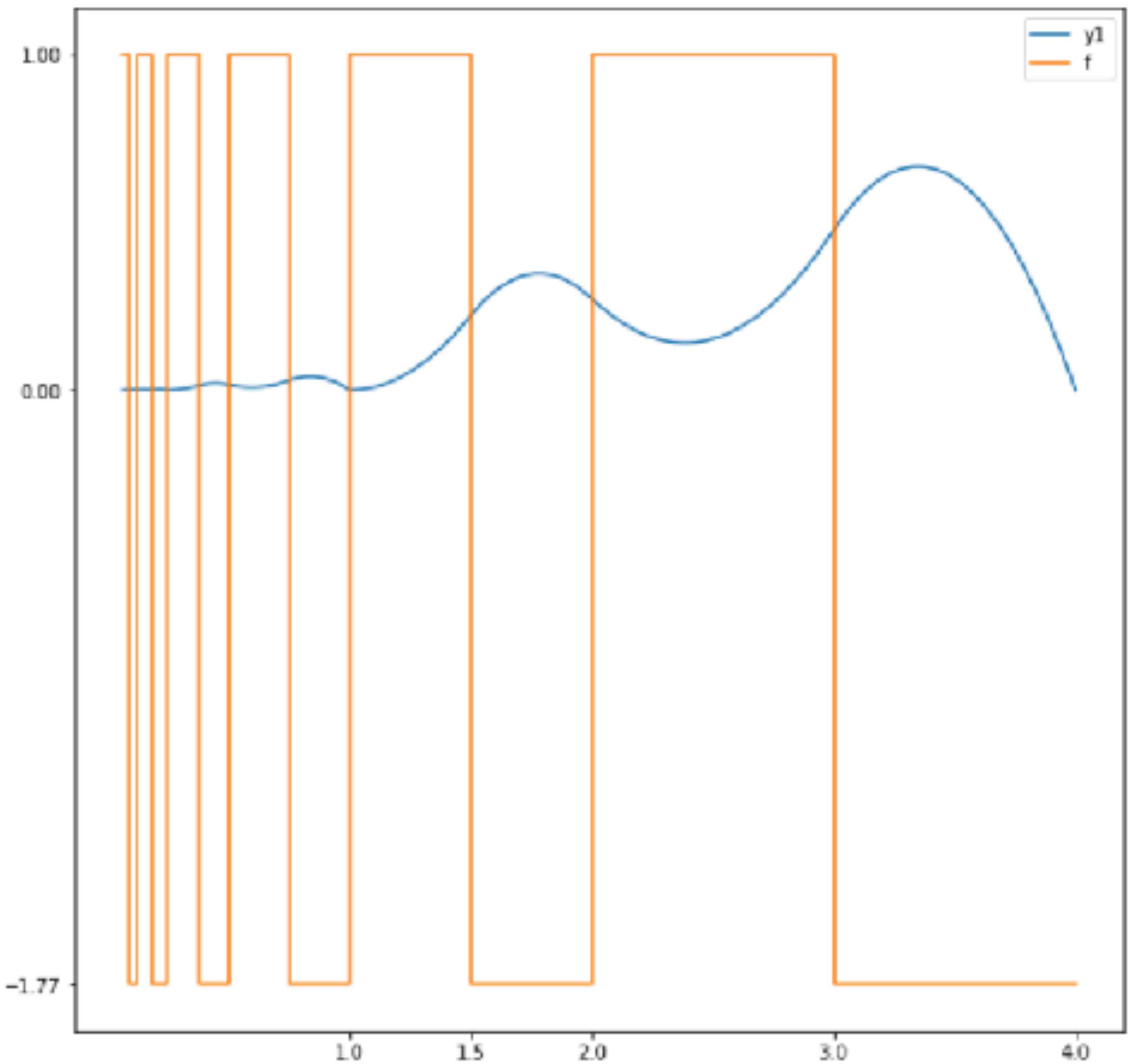
$$q \in W^{1,\infty}, \quad \dot{q} \in BV, \quad \lambda \in \mathcal{M}$$

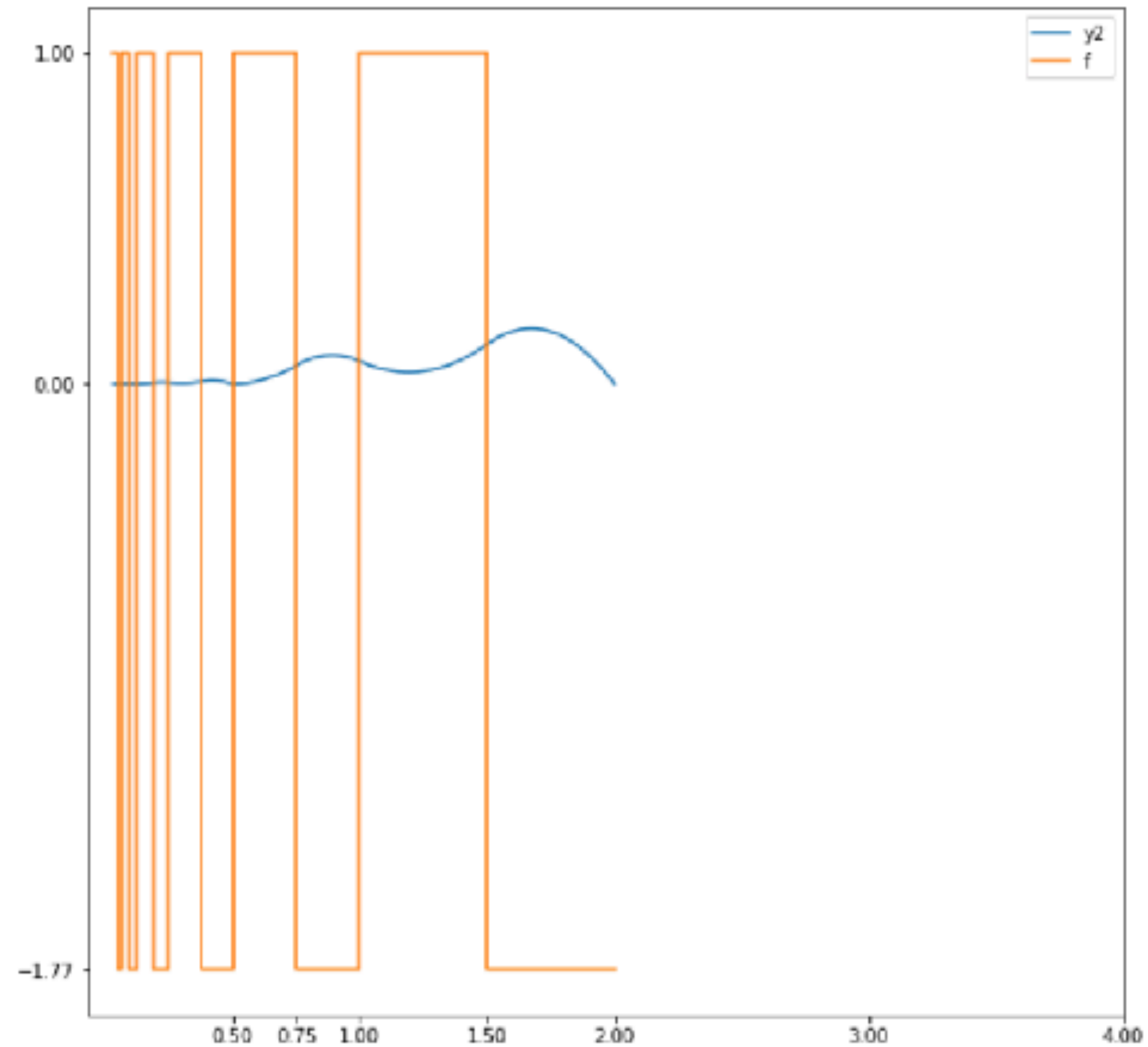
Moreau, J.-J. "Unilateral contact and dry friction in finite freedom dynamics." *Nonsmooth mechanics and Applications* (1988): 1-82.

Non-uniqueness



Non-uniqueness





- Need for analytic forces to obtain uniqueness

- Schatzman, Michelle. "A class of nonlinear differential equations of second order in time." *Nonlinear Analysis: Theory, Methods and Applications* 2.3 (1978): 355-373.
- Ballard, Patrick. "The dynamics of discrete mechanical systems with perfect unilateral constraints." *Archive for Rational Mechanics and Analysis* 154 (2000): 199-274.

Existence

$$\dot{q}^+ = P_{C_q}(\dot{q}^-)$$
$$C_q = \{v \in \mathbb{R} \,/\, v \geq 0 \;\; \text{if} \;\; q = 0\}$$

- Proofs = time discretization + compactness results

$$q^n,\, u^n \;\; \text{given}$$

1) $$\bar{u}^{n+1} = u^n + \frac{\Delta t}{m} f^e(t^n, q^n)$$

2) $$u^{n+1} = P_{m,C_{q^n}}\left(\bar{u}^{n+1}\right)$$
$$C_{q^n} = \{v \in \mathbb{R} \,/\, v \geq 0 \;\; \text{if} \;\; q^n = 0\}$$

3) $$q^{n+1} = q^n + \Delta t\, u^{n+1}$$

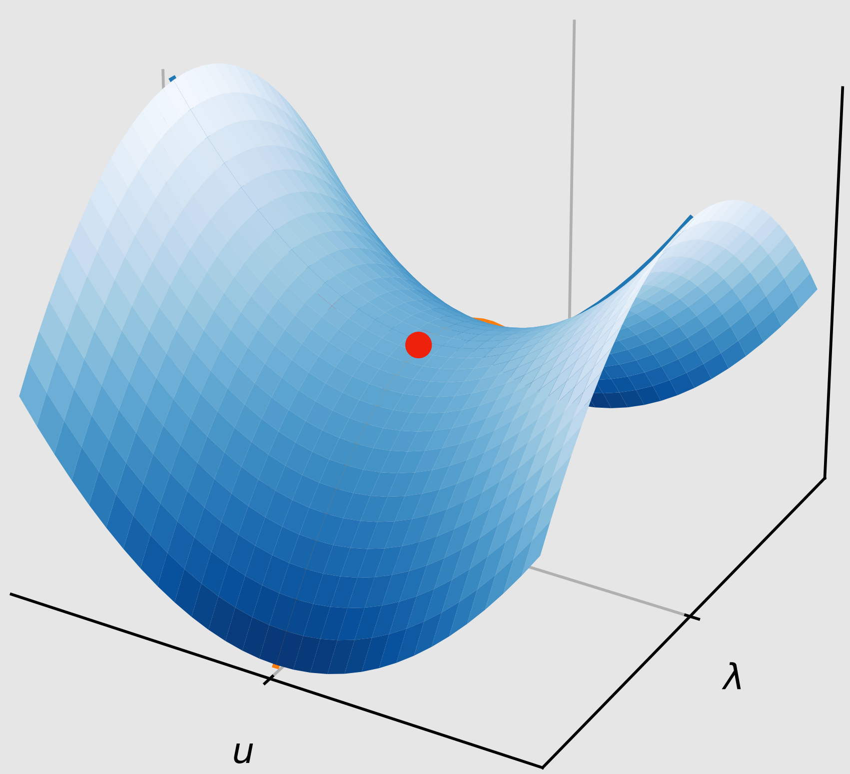
- Paoli, Laetitia, and Michelle Schatzman. "A numerical scheme for impact problems I: The one-dimensional case." *SIAM Journal on Numerical Analysis* 40.2 (2002): 702-733.
- Marques, Manuel DP Monteiro, and Laetitia Paoli. "An existence result in non-smooth dynamics." *Nonsmooth Mechanics and Analysis: Theoretical and Numerical Advances*. Springer US, 2006.

Saddle-points, constrained optimization and duality

$$\mathcal{L} : \mathbf{U} \times \mathbf{\Lambda} \mapsto \mathbb{R} \qquad \mathbf{U} \subset \mathbb{R}^N \qquad \mathbf{\Lambda} \subset \mathbb{R}^m$$

$(u, \lambda) \in \mathbf{U} \times \mathbf{\Lambda}$ is a saddle point of $\mathcal{L}$ if and only if

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda) \quad \forall v \in \mathbf{U}, \mu \in \mathbf{\Lambda}$$



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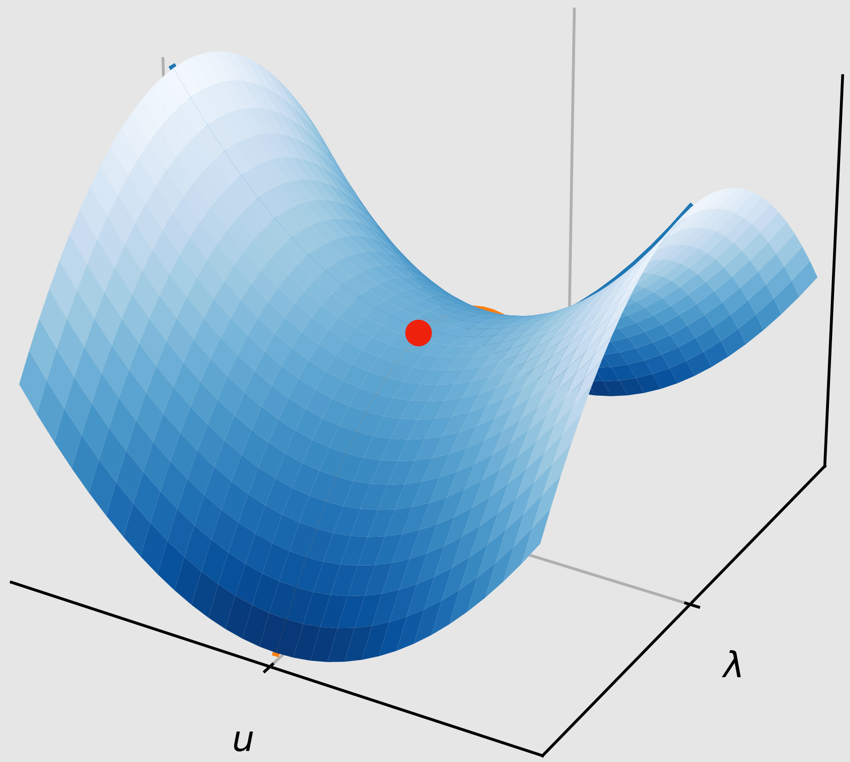
$$\min_{v \in \mathbf{U}} \sup_{\mu \in \mathbf{\Lambda}} \mathcal{L}(v, \mu) = \max_{\mu \in \mathbf{\Lambda}} \inf_{v \in \mathbf{U}} \mathcal{L}(v, \mu)$$



Saddle-points, constrained optimization and duality

$$\left| \begin{array}{l} u \in K \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

where

$$K = \{v \in \mathbb{R}^N,\ g(v) \leq 0\}$$

$$J : \mathbb{R}^N \mapsto \mathbb{R}$$

$$J(v) = \frac{1}{2} a(v, v) - l(v)$$

$$g : \mathbb{R}^N \mapsto \mathbb{R}$$

g affine

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g affine

Lagrangian of the optimization problem:

$$\mathcal{L} : \; \mathbb{R}^N \times \mathbb{R}_+ \mapsto \mathbb{R}$$

$$\mathcal{L}(v,\mu) = J(v) + \mu \, g(v)$$

$$\sup_{\mu \in \mathbb{R}^+} \mathcal{L}(v,\mu) = \left| \begin{array}{ll} J(v) & \text{if } \; v \in K \\ \\ +\infty & \text{else} \end{array} \right.$$

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$$(u, \lambda) \in \mathbb{R}^N \times \mathbb{R}_+$$

(u, λ) saddle-point of $\mathcal{L}$

[if constraints qualified...] $\exists \lambda$ s.t.

Saddle-points, constrained optimization and duality

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$$\min_{v \in \mathbb{R}^N} \sup_{\mu \in \mathbb{R}_+} \mathcal{L}(v, \mu)$$

Primal problem

Saddle-points, constrained optimization and duality

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$$\min_{v \in \mathbb{R}^N} \sup_{\mu \in \mathbb{R}_+} \mathcal{L}(v, \mu)$$

Primal problem

$$\max_{\mu \in \mathbb{R}_+} \inf_{v \in \mathbb{R}^N} \mathcal{L}(v, \mu)$$

Dual problem

Saddle-points, constrained optimization and duality

$$\left| \begin{array}{l} u \in K \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

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$$(u, \lambda) \in \mathbb{R}^N \times \mathbb{R}_+$$

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[if constraints qualified...] $\exists \lambda$ s.t.

$$\min_{v \in \mathbb{R}^N} \sup_{\mu \in \mathbb{R}_+} \mathcal{L}(v, \mu)$$

Primal problem

$$\max_{\mu \in \mathbb{R}_+} \inf_{v \in \mathbb{R}^N} \mathcal{L}(v, \mu)$$

Dual problem

$$\nabla J(u) = -\lambda \nabla g(u)$$

$$g(u) \leq 0, \quad \lambda \geq 0$$

$$g(u)\lambda = 0$$

Optimality conditions

Existence

$$\dot{q}^+ = P_{C_q}(\dot{q}^-)$$

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3) $$q^{n+1} = q^n + \Delta t \, u^{n+1}$$

- Paoli, Laetitia, and Michelle Schatzman. "A numerical scheme for impact problems I: The one-dimensional case." *SIAM Journal on Numerical Analysis* 40.2 (2002): 702-733.
- Marques, Manuel DP Monteiro, and Laetitia Paoli. "An existence result in non-smooth dynamics." *Nonsmooth Mechanics and Analysis: Theoretical and Numerical Advances*. Springer US, 2006.

Existence

$$\dot{q}^+ = P_{C_q}(\dot{q}^-)$$
$$C_q = \{v \in \mathbb{R} \,/\, v \geq 0 \ \text{ if } \ q = 0\}$$

- Proofs = time discretization + compactness results

$$q^n,\, u^n \ \text{ given}$$

1) $$\bar{u}^{n+1} = u^n + \frac{\Delta t}{m} f^e(t^n, q^n)$$

2) $$u^{n+1} = P_{m, C_{q^n}}\left(\bar{u}^{n+1}\right)$$

$$C_{q^n} = \{v \in \mathbb{R} \,/\, v \geq 0 \ \text{ if } \ q^n = 0\}$$

3) $$q^{n+1} = q^n + \Delta t\, u^{n+1}$$

$$\nabla J(u) = -\lambda \nabla g(u)$$

Optimality conditions

$$\exists \lambda^{n+1} \geq 0 \ \text{ s.t. } \quad m \frac{u^{n+1} - u^n}{\Delta t} = f^e(t^n, q^n) + \lambda^{n+1}$$

Numerical schemes

- Algorithms based on the discretization used to prove existence
  - Explicit admissible velocity space

$$C_{q^n} = \{v \in \mathbb{R} \,/\, v \geq 0 \;\text{ if }\; q^n \leq 0\}$$

  - Implicit contact forces

Find (u^{n+1}, λ^{n+1}) with $m\dfrac{u^{n+1} - u^n}{\Delta t} = f^e(t^n, q^n) + \lambda^{n+1}$

- Can lead to non-admissible configurations (overlapping)

Numerical schemes

$$\left| \begin{array}{l} u \in K \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

Optimization problem

- Fully implicit algorithms

  - Implicit admissible velocity space

$$C_{q^n} = \{v \in \mathbb{R} \,/\, q^{n+1} = q^n + \Delta t\, v \geq 0\}$$

  - Implicit contact forces

Find (u^{n+1}, λ^{n+1}) with $m\dfrac{u^{n+1} - u^n}{\Delta t} = f^e(t^n, q^n) + \lambda^{n+1}$

- Admissible configurations, large time-steps

- Anitescu, M., & Hart, G. D. (2004). A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction. *International Journal for Numerical Methods in Engineering*, *60*(14), 2335-2371.
- B. Maury. A time-stepping scheme for inelastic collisions. *Numerische Mathematik*, 102(4):649–679, 2006.

Numerical schemes

$$\nabla J(u) = -\lambda \nabla g(u)$$
$$g(u) \leq 0, \quad \lambda \geq 0$$
$$g(u)\lambda = 0$$

Optimality conditions

- Fully implicit algorithms - Convergence result

Model

$$m\ddot{q} = f^e + \lambda$$

$$q \geq 0, \quad \lambda \geq 0$$
$$\mathrm{supp}(\lambda) \subset \{t \,/\, q(t) = 0\}$$

$$\dot{q}^+ = P_{C_q}(\dot{q}^-)$$
$$C_q = \{v \in \mathbb{R} \,/\, v \geq 0 \;\text{ if }\; q = 0\}$$

Algorithm

$$m\frac{u^{n+1} - u^n}{\Delta t} = f^e(t^n, q^n) + \lambda^{n+1}$$

$$q^n + \Delta t\, u^{n+1} \geq 0, \quad \lambda^{n+1} \geq 0$$
$$\left(q^n + \Delta t\, u^{n+1}\right)\, \lambda^{n+1} = 0$$

$$u^{n+1} = P_{m, C_{q^n}}\left(\bar{u}^{n+1}\right)$$
$$C_{q^n} = \{v \in \mathbb{R} \,/\, q^{n+1} = q^n + \Delta t\, v \geq 0\}$$

- B. Maury. A time-stepping scheme for inelastic collisions. *Numerische Mathematik*, 102(4):649–679, 2006.

Numerical schemes

$$\max_{\mu \in \mathbb{R}_+} \inf_{v \in \mathbb{R}^N} \mathcal{L}(v, \mu)$$

Dual problem

- Fully implicit algorithms - Dual problem

Numerical schemes

$$\max_{\mu \in \mathbb{R}_+} \inf_{v \in \mathbb{R}^N} \mathcal{L}(v, \mu)$$

Dual problem

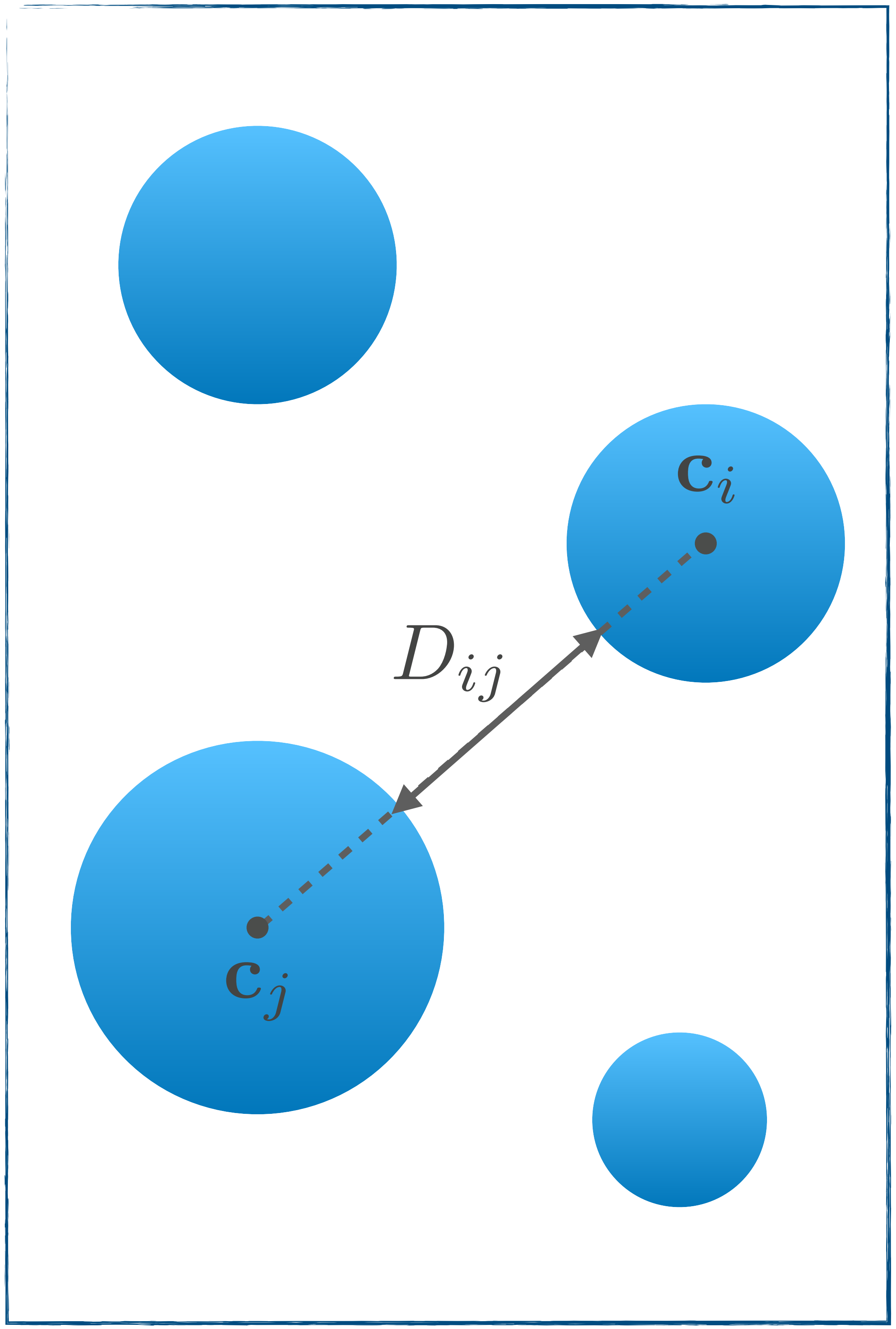
- Fully implicit algorithms - Dual problem

$$\min_{\mu \in \mathbb{R}^+} \mu \left(q^n + \Delta t \, \frac{u_\mu + \bar{u}^{n+1}}{2} \right)$$

where $$m \frac{u_\mu - u^n}{\Delta t} = f^e(t^n, q^n) + \mu$$

- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication
- M. Frémond. Non-Smooth Thermomechanics. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002

The multi particle case



$$\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_N) \in \mathbb{R}^{3N} \qquad \mathbf{v} = \dot{\mathbf{c}}$$

$$M\ddot{\mathbf{x}} = \mathbf{f}^e + \sum_{i<j} \lambda_{ij} \nabla D_{ij}(\mathbf{c})$$

$$\mathrm{supp}(\lambda_{ij}) \subset \{t \,/\, D_{ij}(\mathbf{c}(t)) = 0\}$$

$$D_{ij}(\mathbf{c}(t)) \geq 0, \quad \lambda_{ij} \geq 0$$

$$\dot{\mathbf{c}}^+ = P_{C_\mathbf{c}}(\dot{\mathbf{c}}^-)$$

$$C_\mathbf{c} = \{\mathbf{v} \in \mathbb{R}^{3N} \,/\, \forall i < j, \; \nabla D_{ij} \cdot \mathbf{v} \geq 0 \text{ if } D_{ij}(\mathbf{c}) = 0\}$$

The multi particle case

- Fully implicit algorithm

$$\mathbf{c}^n, \mathbf{v}^n \text{ given}$$

1) $$\bar{\mathbf{v}}^{n+1} = \mathbf{v}^n + \Delta t\, M^{-1} \mathbf{f}^e(t^n, \mathbf{c}^n)$$

2) $$\mathbf{v}^{n+1} = P_{M,C^n}(\bar{\mathbf{v}}^{n+1})$$

$$C^n = \{\mathbf{v} \in \mathbb{R}^{3N} \,/\, \forall i < j, \quad D_{ij}(\mathbf{c}^n) + \Delta t \nabla D_{ij}(\mathbf{c}^n) \cdot \mathbf{v} \geq 0\}$$

3) $$q^{n+1} = q^n + \Delta t\, u^{n+1}$$

The multi particle case

- Fully implicit algorithm

$$\mathbf{c}^n, \mathbf{v}^n \text{ given}$$

1) $\bar{\mathbf{v}}^{n+1} = \mathbf{v}^n + \Delta t\, M^{-1} \mathbf{f}^e(t^n, \mathbf{c}^n)$

2) $\mathbf{v}^{n+1} = P_{M,C^n}(\bar{\mathbf{v}}^{n+1})$

$$C^n = \{\mathbf{v} \in \mathbb{R}^{3N} \,/\, \forall i < j, \; D_{ij}(\mathbf{c}^n) + \Delta t \nabla D_{ij}(\mathbf{c}^n) \cdot \mathbf{v} \geq 0\}$$

3) $q^{n+1} = q^n + \Delta t\, u^{n+1}$

$$\nabla J(\mathbf{v}) = -\sum_{i<j} \lambda_{ij} \nabla g_{ij}(\mathbf{v})$$

Optimality conditions

$$\exists \lambda_{ij} \geq 0 \text{ s.t. } M \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = \mathbf{f}^e(t^n, \mathbf{c}^n) + \sum_{i<j} \lambda_{ij} \nabla D_{ij}(\mathbf{c}^n)$$

The multi particle case

- Fully implicit algorithm

$$\mathbf{c}^n, \mathbf{v}^n \text{ given}$$

1) $\bar{\mathbf{v}}^{n+1} = \mathbf{v}^n + \Delta t\, M^{-1}\mathbf{f}^e(t^n, \mathbf{c}^n)$

2) $\mathbf{v}^{n+1} = P_{M,C^n}(\bar{\mathbf{v}}^{n+1})$

$$C^n = \{\mathbf{v} \in \mathbb{R}^{3N} \,/\, \forall i < j, \quad D_{ij}(\mathbf{c}^n) + \Delta t \nabla D_{ij}(\mathbf{c}^n) \cdot \mathbf{v} \geq 0\}$$

3) $q^{n+1} = q^n + \Delta t\, u^{n+1}$

- B. Maury. A time-stepping scheme for inelastic collisions. *Numerische Mathematik*, 102(4):649–679, 2006.
- Bernicot, F., & Lefebvre-Lepot, A. (2010). Existence results for nonsmooth second-order differential inclusions, convergence result for a numerical scheme and application to the modeling of inelastic collisions. *Confluentes Mathematici*, *2*(04), 445-471.

The multi particle case

- Fully implicit algorithm - the corresponding dual problem

$$\min_{\lambda_{ij}\in\mathbb{R}^+} \sum_{i<j} \lambda_{ij} \left[D_{ij}(\mathbf{c}^n) + \Delta t\, \mathbf{n}_{ij} \cdot \mathbf{A}_{ij} \left(\frac{\mathbf{v}_{\boldsymbol{\lambda}} + \bar{\mathbf{v}}^{n+1}}{2} \right) \right]$$

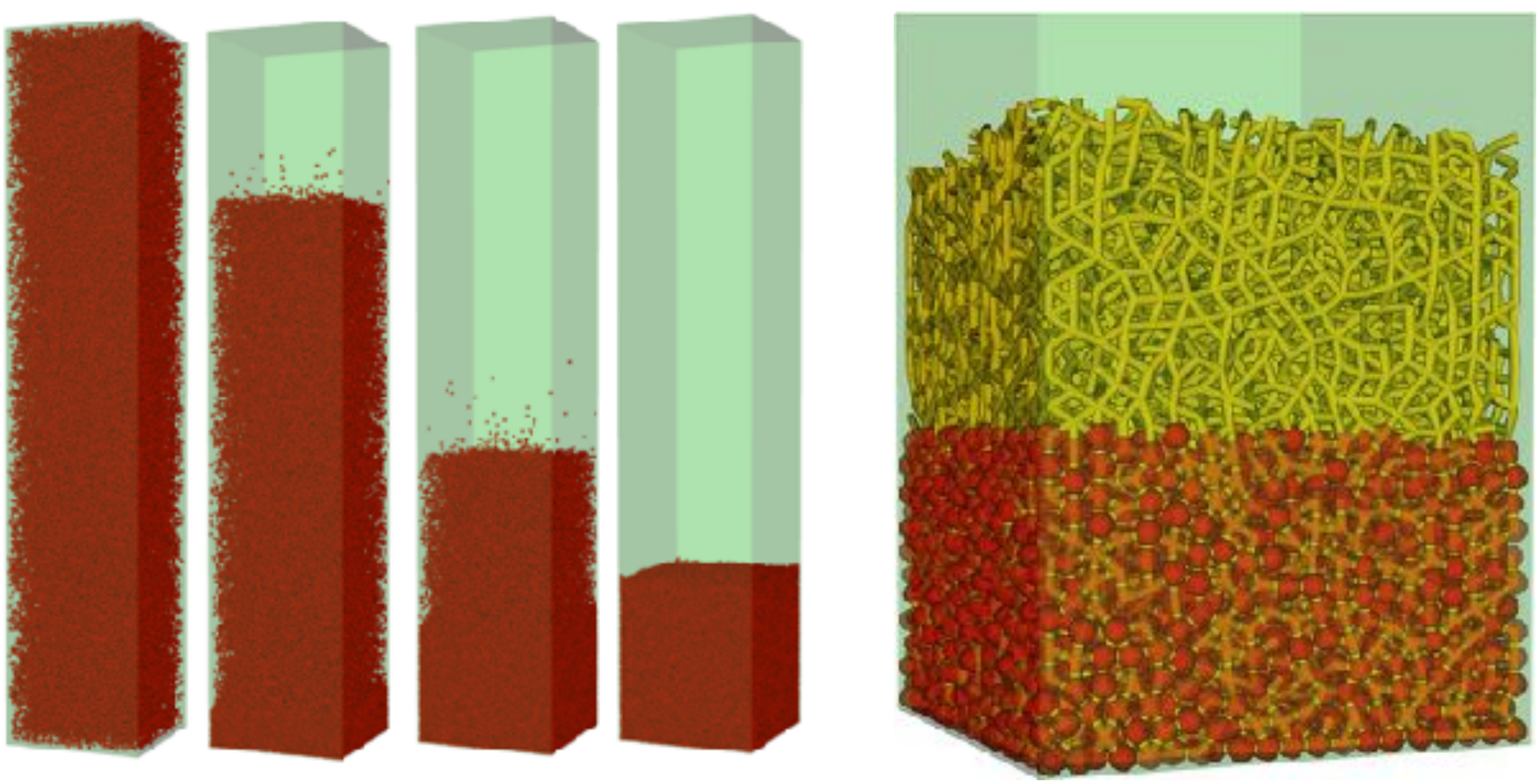
where $$M \frac{\mathbf{v}_{\boldsymbol{\lambda}} - \mathbf{v}^n}{\Delta t} = \mathbf{f}^e(t^n, \mathbf{c}^n) + \sum_{i<j} \lambda_{ij} \nabla D_{ij}(\mathbf{c}^n)$$

- Can be solved using Projected Gradient Algorithms

- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

Application: random packing of non-convex particles

- Inelastic contacts
- SCoPI
- Validation for spheres

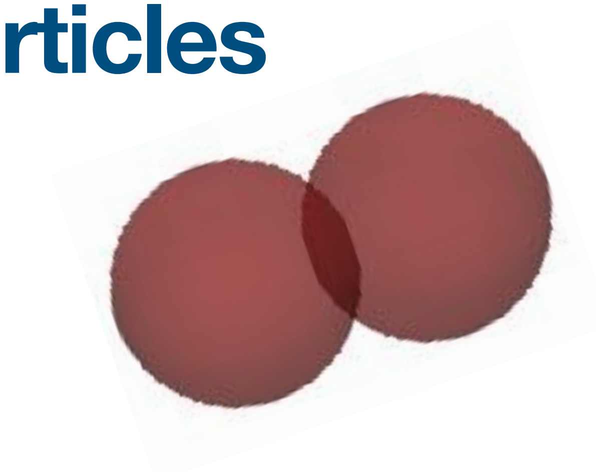


Random packing:

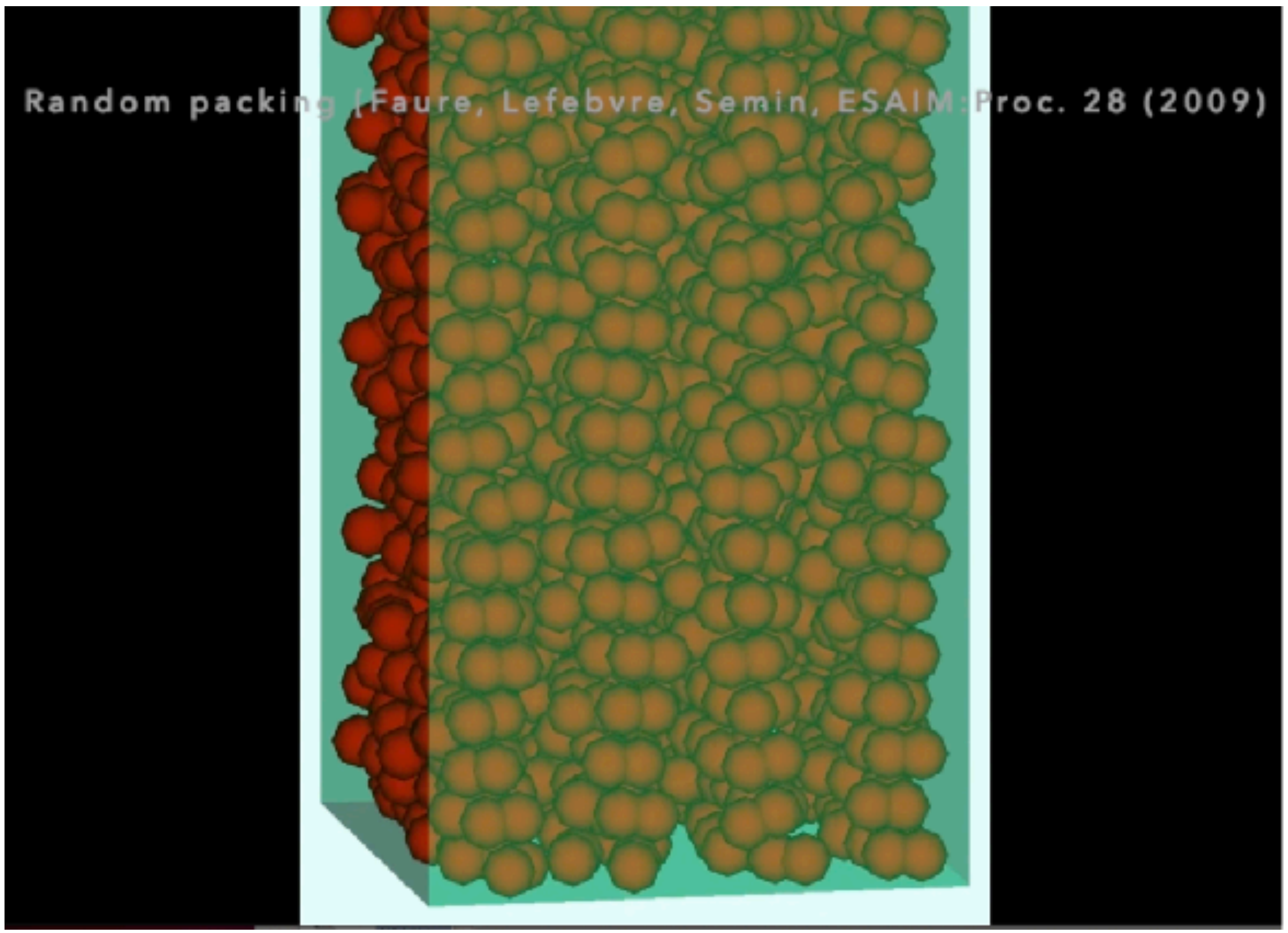
$63.7 \pm 0.2\%$

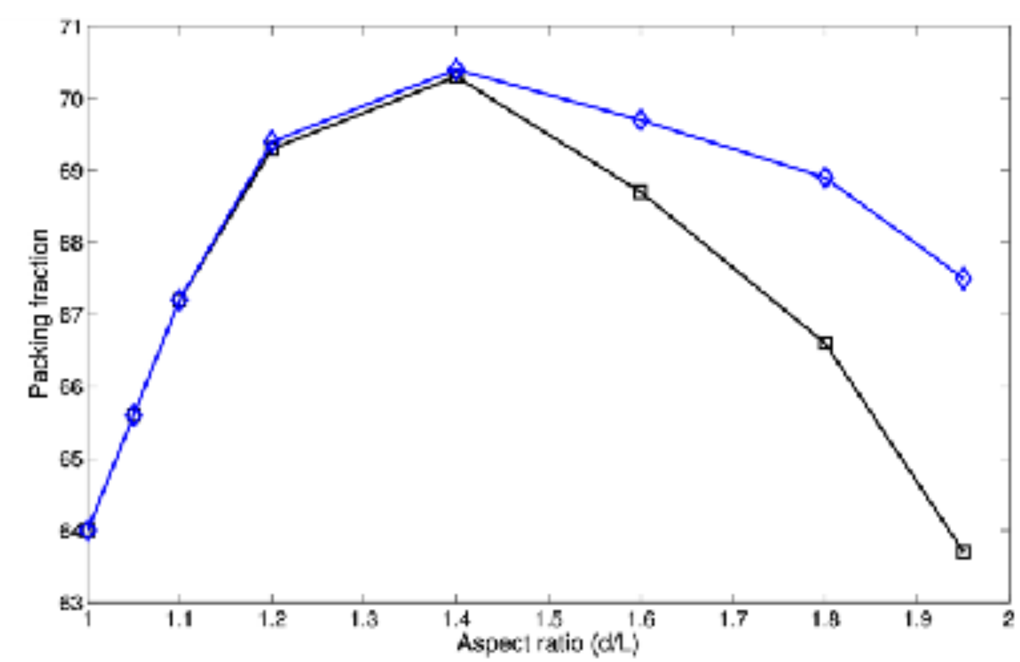
Faure, S., Lefebvre-Lepot, A., & Semin, B. (2009). Dynamic numerical investigation of random packing for spherical and nonconvex particles. In *Esaim: Proceedings* (Vol. 28, pp. 13-32). EDP Sciences.

Application: random packing of non-convex particles



- Inelastic contacts
- SCoPI
    - Case of non-convex particles

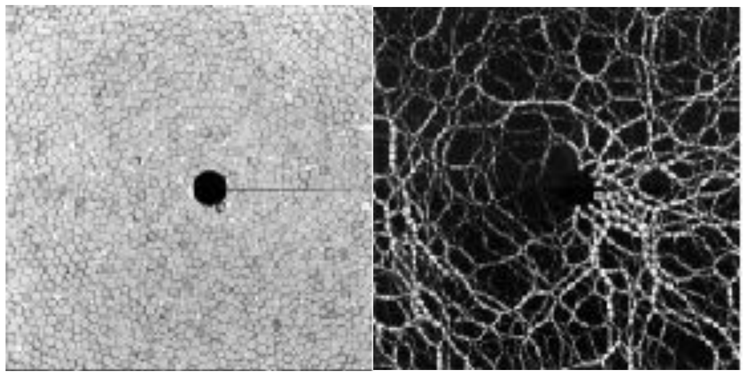


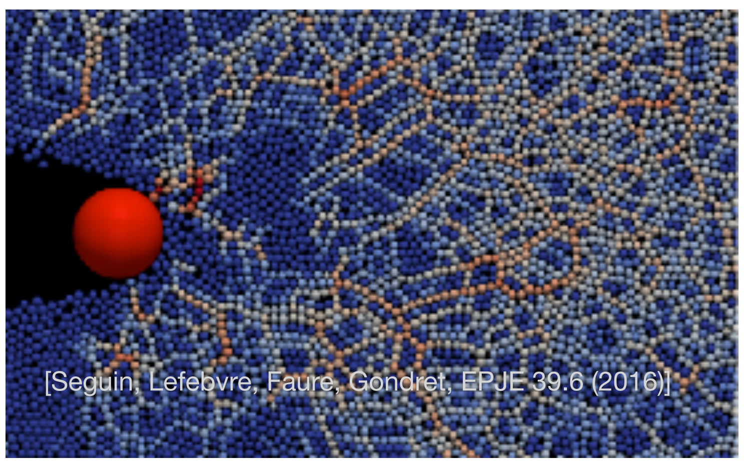


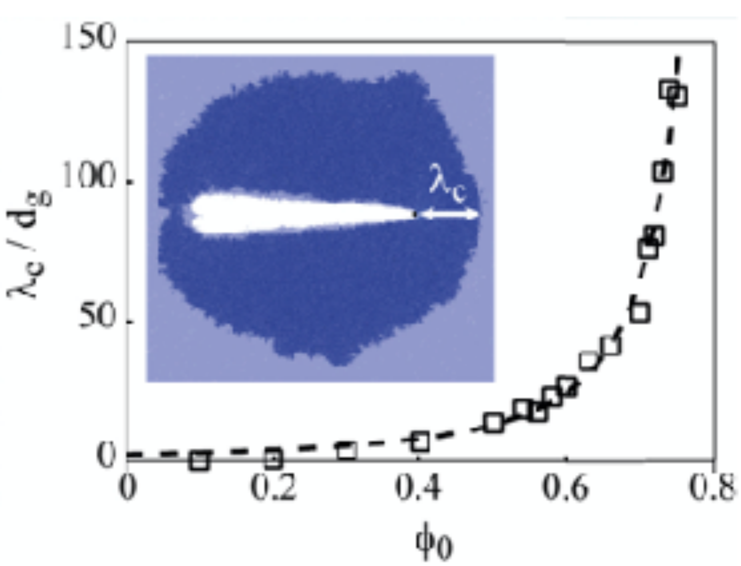
Faure, S., Lefebvre-Lepot, A., & Semin, B. (2009). Dynamic numerical investigation of random packing for spherical and nonconvex particles. In *Esaim: Proceedings* (Vol. 28, pp. 13-32). EDP Sciences.

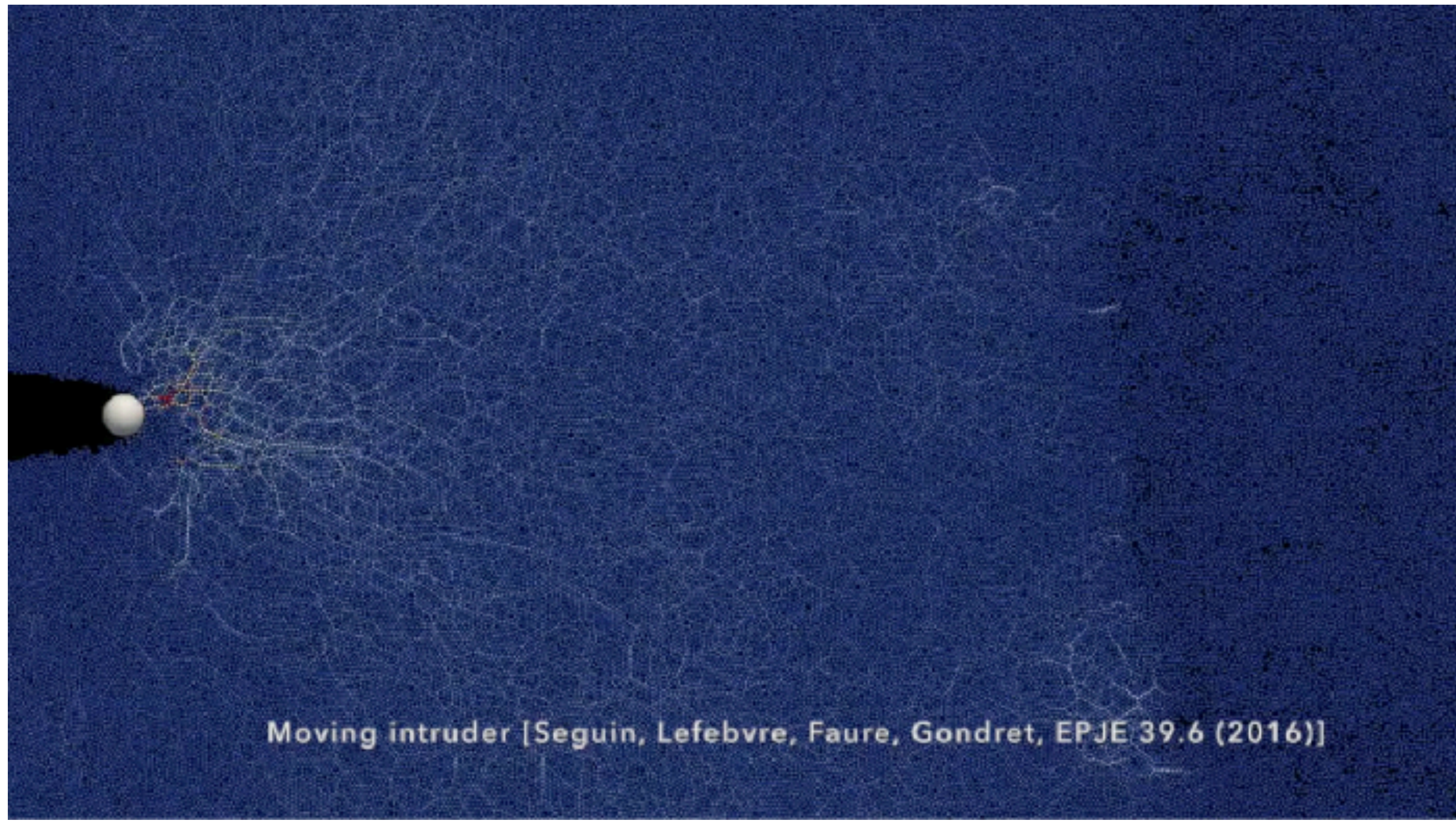
Application: clustering and flow around an intruder

- Inelastic contacts
- Spherical particles
- SCoPI







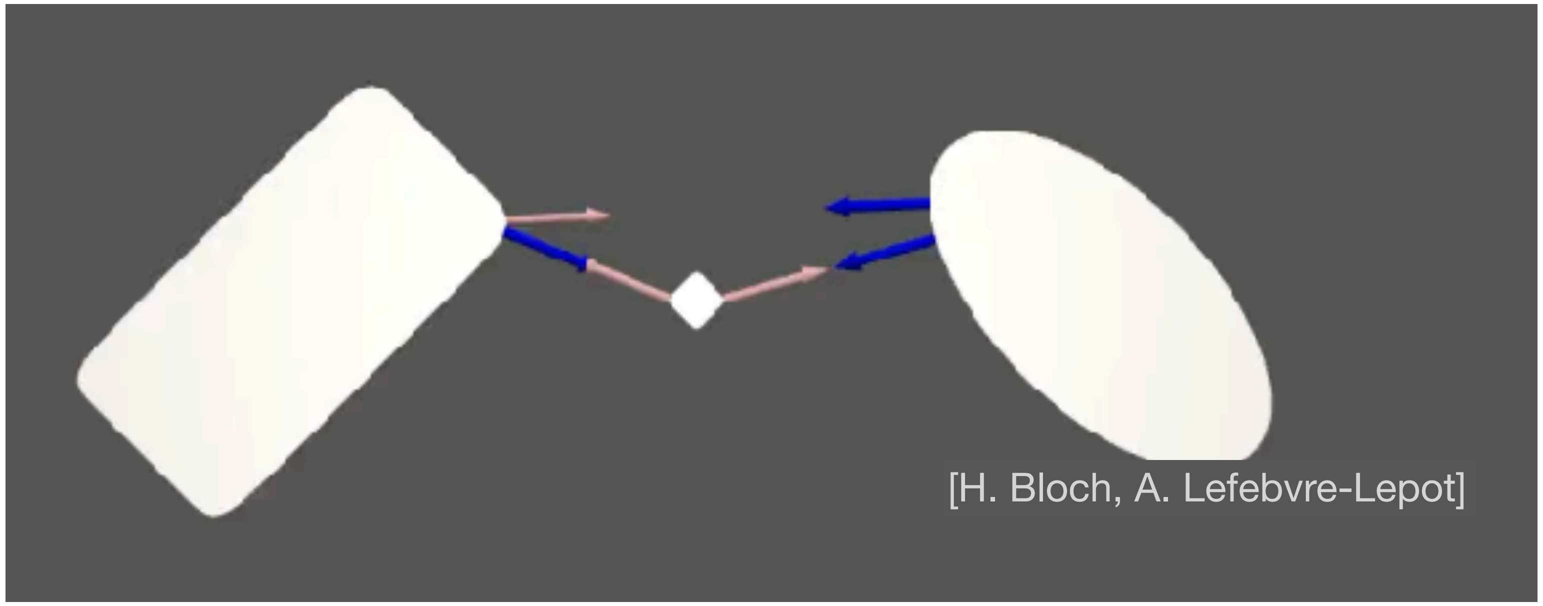


- Rheology
- Chains of forces

Clustering and flow around a sphere moving into a grain cloud. With A. Seguin, S. Faure, and P. Gondret.
In: The European Physical Journal E 39.6 (2016)

Extension to non-spherical particles.

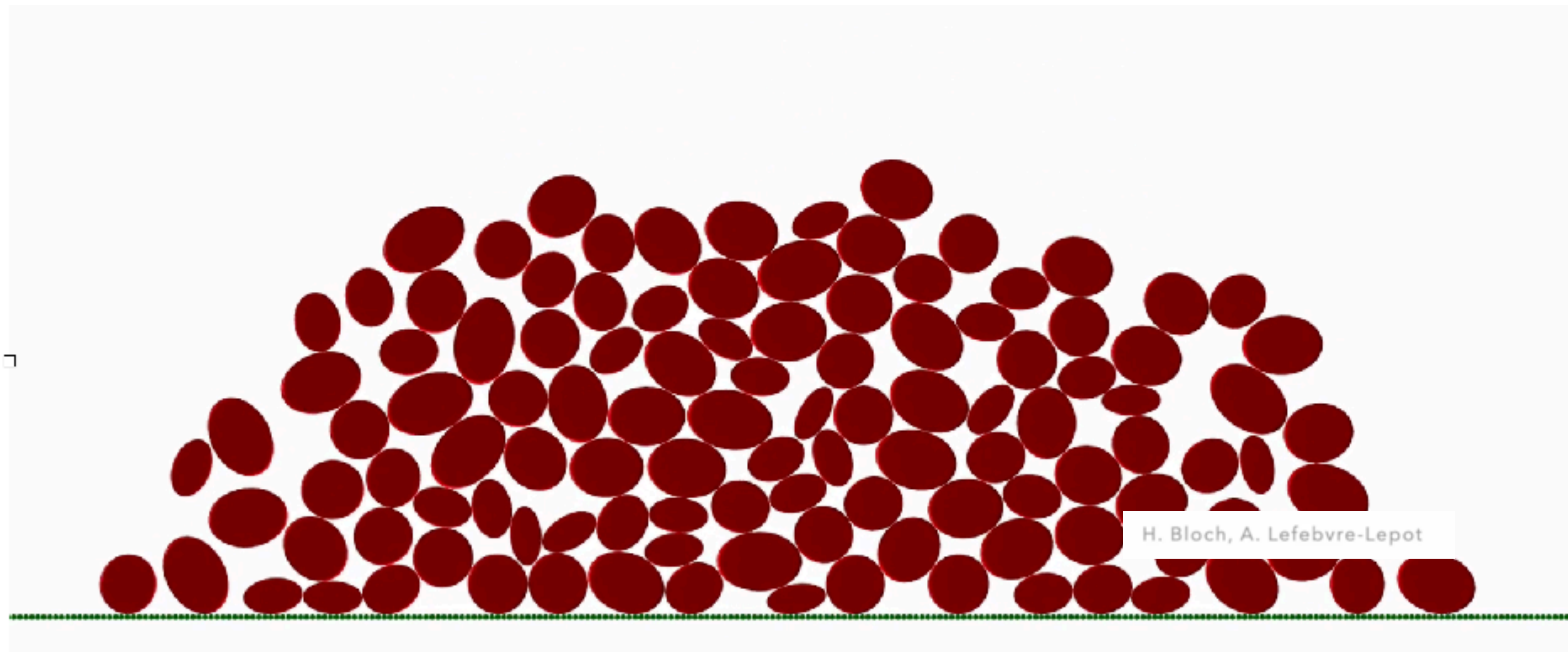
- Non-spherical particles
- Hélène Bloch [CMAP]
- SCoPI



On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

Stable and convex contact algorithms.

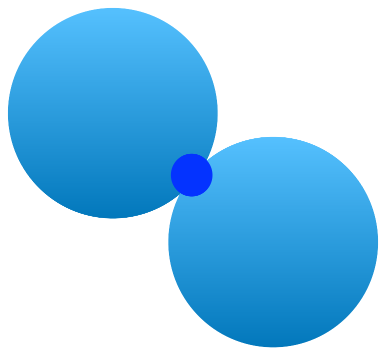
- Non-spherical particles
- Hélène Bloch [CMAP]
- SCoPI



On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

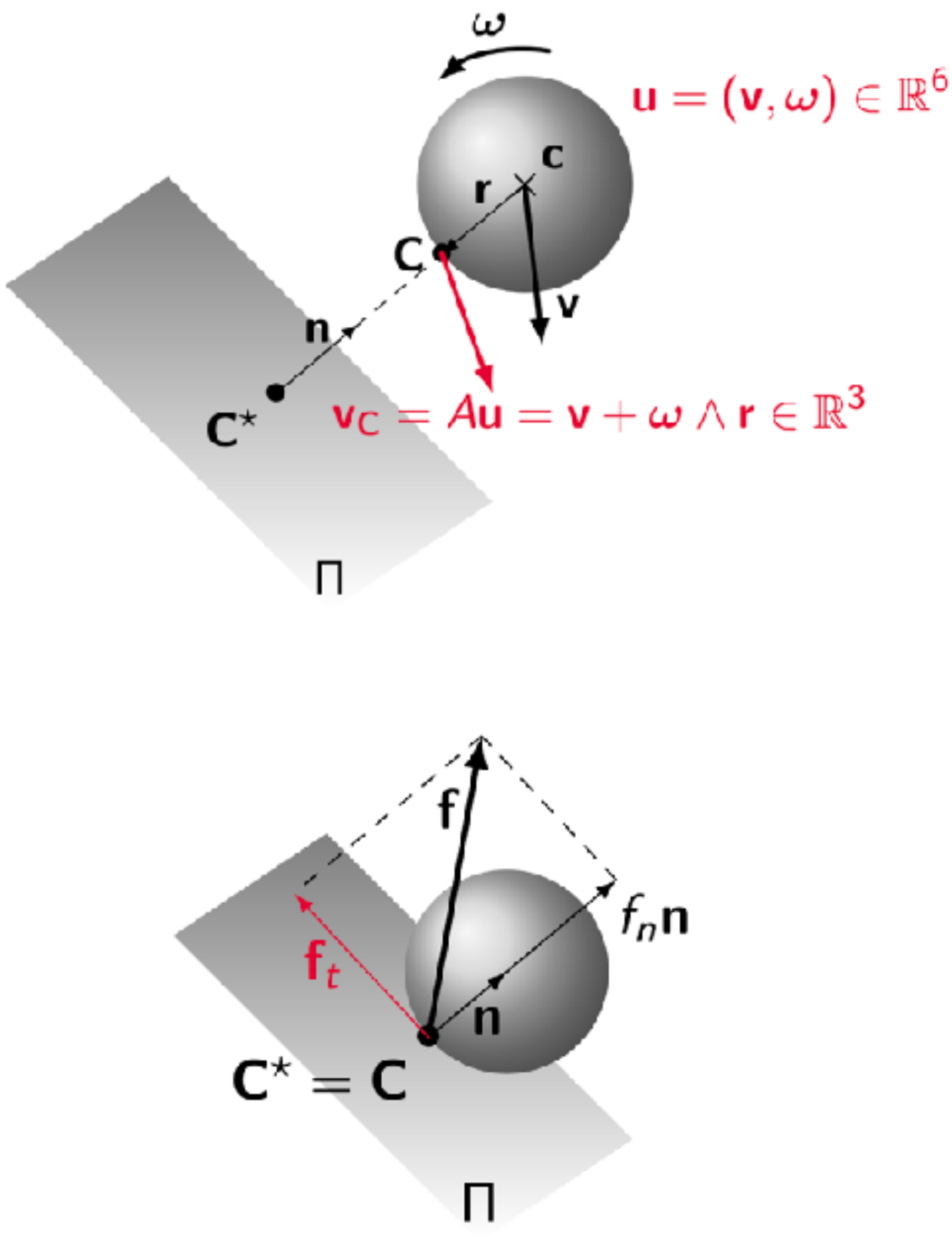
Contact Dynamics Models including friction

Contact law



Inelastic contacts
Coulomb friction law

Modeling friction.



- Contact force

$$\mathbf{f} = f_n \mathbf{n} + \mathbf{f}_t$$

- If $P\mathbf{v_C}^+ = 0$ (no slip)

$$|\mathbf{f}_t| \leq \mu f_n$$

- If $P\mathbf{v_C}^+ \neq 0$ (sliding motion)

$$\mathbf{f}_t = -\mu f_n \frac{P\mathbf{v_C}^+}{|P\mathbf{v_C}^+|}$$

Modeling friction.

$$M\frac{d\mathbf{u}}{dt} = \mathbf{f}^e + A^T(f_n\mathbf{n} + \mathbf{f}_t) \quad \text{(FPD)}$$

$$D(\mathbf{c}) \geq 0, \quad f_n \geq 0, \quad D(\mathbf{c})f_n = 0 \quad \text{(Norm. Cont.)}$$

$$\mathbf{u}^+ = P_{C_\mathbf{c}}\mathbf{u}^-$$

If $PA\mathbf{u}^+ = 0$ (no slip), $|\mathbf{f}_t| \leq \mu f_n$ (Tang. Cont.)

If $PA\mathbf{u}^+ \neq 0$ (sliding motion), $\mathbf{f}_t = -\mu f_n \frac{PA\mathbf{u}^+}{|PA\mathbf{u}^+|}$

$$\left[\mathbf{u} = (\mathbf{v}, \boldsymbol{\omega}), \quad A\mathbf{u} = \mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{r}, \quad A^T\mathbf{f} = (\mathbf{f}, \mathbf{r} \wedge \mathbf{f}) \in \mathbb{R}^6\right]$$

Moreau, J.-J. "Unilateral contact and dry friction in finite freedom dynamics." *Nonsmooth mechanics and Applications* (1988): 1-82.

Founding NSCD algorithms

$$M\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t}=\mathbf{f}^{e}+A^{n,T}(f_n\mathbf{n}^{n}+\mathbf{f}_t)$$

(FPD)

$$f_n \geq 0$$

$$D^n + \Delta t\,\nabla_{\mathbf{c}} D^n \cdot \mathbf{v}^{n+1} \geq 0$$

$$(D^n + \Delta t\,\nabla_{\mathbf{c}} D^n \cdot \mathbf{v}^{n+1}) f_n = 0$$

(Norm. Cont.)

If $P^n A^n \mathbf{u}^{n+1} = 0$ $\quad |\mathbf{f}_t| \leq \mu f_n$

If $P^n A^n \mathbf{u}^{n+1} \neq 0$ $\quad \mathbf{f}_t = -\mu f_n \dfrac{P^n A^n \mathbf{u}^{n+1}}{|P^n A^n \mathbf{u}^{n+1}|}$

(Tang. Cont.)

- Jean, M. (1999). The non-smooth contact dynamics method. *Computer methods in applied mechanics and engineering*, *177*(3-4), 235-257.
- Dubois, F., & Jean, M. (2006). The non smooth contact dynamic method: recent LMGC90 software developments and application. *Analysis and simulation of contact problems*, 375-378.

Convex optimization for friction?

Lagrange multipliers

$$\left| \begin{array}{l} u \in K \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

$$K = \{v \in \mathbb{R}^N,\ g(v) \leq 0\}$$

Under qualification conditions,

$\exists \lambda \in \mathbb{R}$ such that

$$\nabla J(u) = -\lambda \nabla g(u)$$

$$g(u) \leq 0, \quad \lambda \geq 0$$

$$g(u)\lambda = 0$$

Optimality conditions

$$M \frac{d\mathbf{u}}{dt} = \mathbf{f}^e + A^T(f_n \mathbf{n} + \mathbf{f}_t)$$

$$D(\mathbf{c}) \geq 0, \quad f_n \geq 0, \quad D(\mathbf{c}) f_n = 0$$

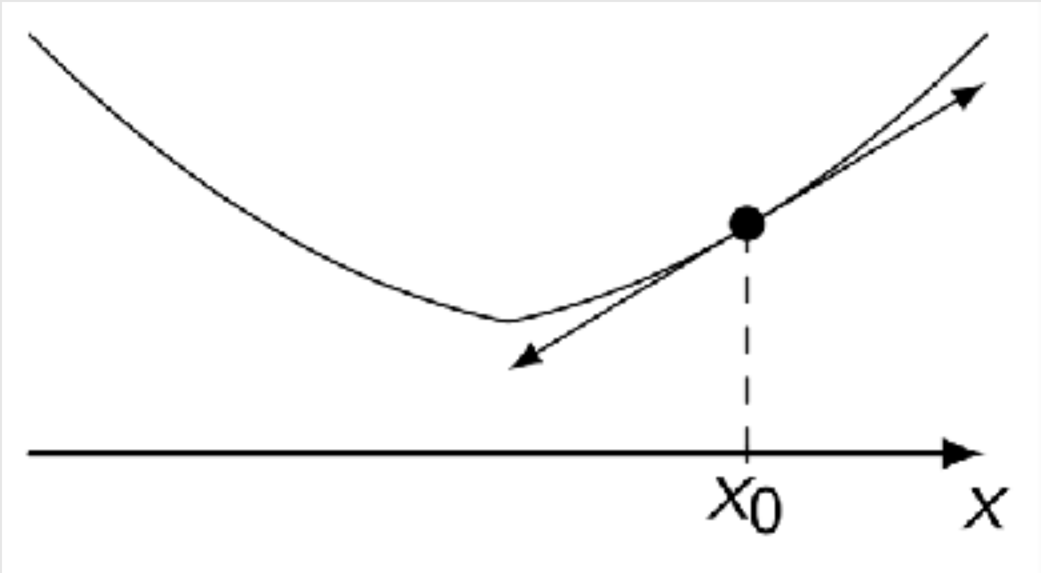
$$\mathbf{u}^+ = P_{C_\mathbf{c}} \mathbf{u}^-$$

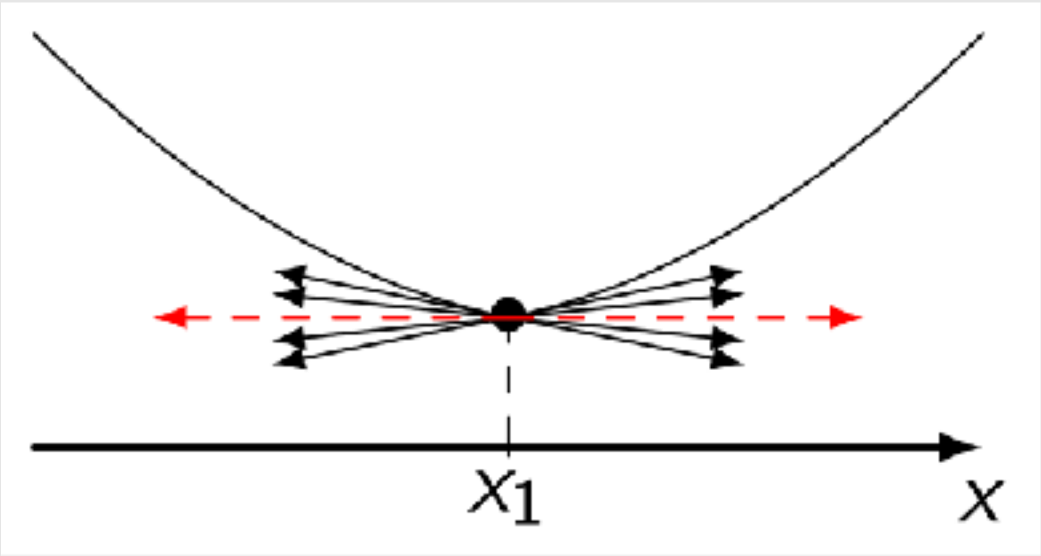
$$PA\mathbf{u}^+ \neq 0 \quad \mathbf{f}_t = -\mu f_n \frac{PA\mathbf{u}^+}{|PA\mathbf{u}^+|}$$

$$PA\mathbf{u}^+ = 0 \quad |\mathbf{f}_t| \leq \mu f_n$$

$\Rightarrow$ Need for non-derivable g

Notion of sub-differential





Sub-differential of a convex function

$\Phi : \mathbb{R}^n \to \mathbb{R}$, convex function

$$\partial\Phi[\mathbf{x}] = \{\mathbf{y} \in \mathbb{R}^n \,/\, \forall \hat{\mathbf{x}} \in \mathbb{R}^n, \;\; \Phi(\hat{\mathbf{x}}) \geq \Phi(\mathbf{x}) + \mathbf{y} \cdot (\hat{\mathbf{x}} - \mathbf{x})\}$$

Optimality condition: $\mathbf{x}$ minimum of $\Phi \iff 0 \in \partial\Phi[\mathbf{x}]$

Notion of sub-differential

- Lagrange multiplier

$$\begin{cases} u \in K \\ J(u) = \min_{v \in K} J(v) \end{cases} \qquad K = \{v \in \mathbb{R}^N,\ g(v) \leq 0\}$$

Under qualification conditions, $\exists \lambda \in \mathbb{R}$ such that

$$\nabla J(u) \in -\lambda\, \partial g[u]$$

$$g(u) \leq 0, \quad \lambda \geq 0$$

$$g(u)\lambda = 0$$

Optimality conditions

- J-B. Hiriart-Urruty and C. Lemarechal. Convex Analysis and Minimization Algorithms I: Fundamentals. Grundlehren der mathematischen Wissenschaften, Convex Analysis and Minimization Algorithms. Springer-Verlag, Berlin Heidelberg, 1993. [Chap. VI]

Convex optimization for friction?

Lagrange multipliers

$$\left| \begin{array}{l} u \in K \\ \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

$$K = \{v \in \mathbb{R}^N, \; g(v) \leq 0\}$$

Under qualification conditions,

$\exists \lambda \in \mathbb{R}$ such that

$$\nabla J(u) \in -\lambda \partial g[u]$$

$$g(u) \leq 0, \quad \lambda \geq 0$$

$$g(u)\lambda = 0$$

$$M \frac{d\mathbf{u}}{dt} = \mathbf{f}^e + A^T(f_n \mathbf{n} + \mathbf{f}_t)$$

$$D(\mathbf{c}) \geq 0, \quad f_n \geq 0, \quad D(\mathbf{c}) f_n = 0$$

$$\mathbf{u}^+ = P_{C_\mathbf{c}} \mathbf{u}^-$$

$$PA\mathbf{u}^+ \neq 0 \quad \mathbf{f}_t = -\mu f_n \frac{PA\mathbf{u}^+}{|PA\mathbf{u}^+|}$$

$$PA\mathbf{u}^+ = 0 \quad |\mathbf{f}_t| \leq \mu f_n$$

$\Rightarrow$ Need for non-derivable g

Convex optimization for friction?

- Fully implicit algorithm

$\mathbf{c}^n, \mathbf{u}^n \;\; \text{given}$

1) $\bar{\mathbf{u}}^{n+1} = \mathbf{u}^n + \Delta t\, M^{-1} \mathbf{f}^e(t^n, \mathbf{c}^n)$

2) $\mathbf{u}^{n+1} = P_{M, C_\mu^n}(\bar{\mathbf{u}}^{n+1})$

$$C_\mu^n = \{\mathbf{v} \in \mathbb{R}^6 \,/\, D(\mathbf{c}^n) + \Delta t\, \nabla D(\mathbf{c}^n) \cdot \mathbf{v} \geq \;\mu \Delta t \,|P^n A^n \mathbf{u}|\,\}$$

3) $\mathbf{c}^{n+1} = \mathbf{c}^n + \Delta t\, \mathbf{v}^{n+1}$

Convex optimization for friction?

- Fully implicit algorithm

$$M\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} = \mathbf{f}^e + A^{n,T}(f_n\mathbf{n}^n + \mathbf{f}_t) \qquad \text{(FPD)}$$

$$f_n \geq 0 \qquad \text{(Norm. Cont.)}$$

$$D^n + \Delta t \nabla_{\mathbf{c}} D^n \cdot \mathbf{v}^{n+1} \geq \mu \Delta t\,|P^n A^n \mathbf{u}^{n+1}|$$

$$(D^n + \Delta t\, \nabla_{\mathbf{c}} D^n \cdot \mathbf{v}^{n+1} - \mu\Delta t\,|P^n A^n \mathbf{u}^{n+1}|)\, f_n = 0$$

$$\text{If } P^n A^n \mathbf{u}^{n+1} = 0 \qquad |\mathbf{f}_t| \leq \mu f_n \qquad \text{(Tang. Cont.)}$$

$$\text{If } P^n A^n \mathbf{u}^{n+1} \neq 0 \qquad \mathbf{f}_t = -\mu f_n \frac{P^n A^n \mathbf{u}^{n+1}}{|P^n A^n \mathbf{u}^{n+1}|}$$

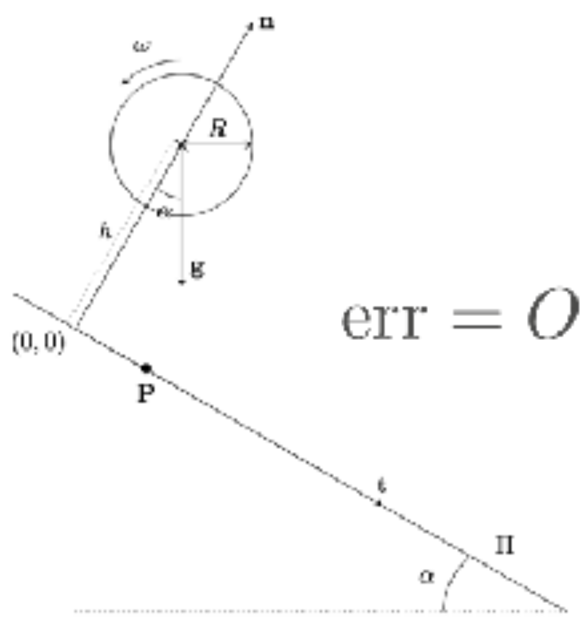
- Anitescu, M. (2006). Optimization-based simulation of nonsmooth rigid multibody dynamics. *Mathematical Programming*, *105*, 113-143.
- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

Convex optimization for friction?

- Fully implicit algorithm

- Order of convergence?

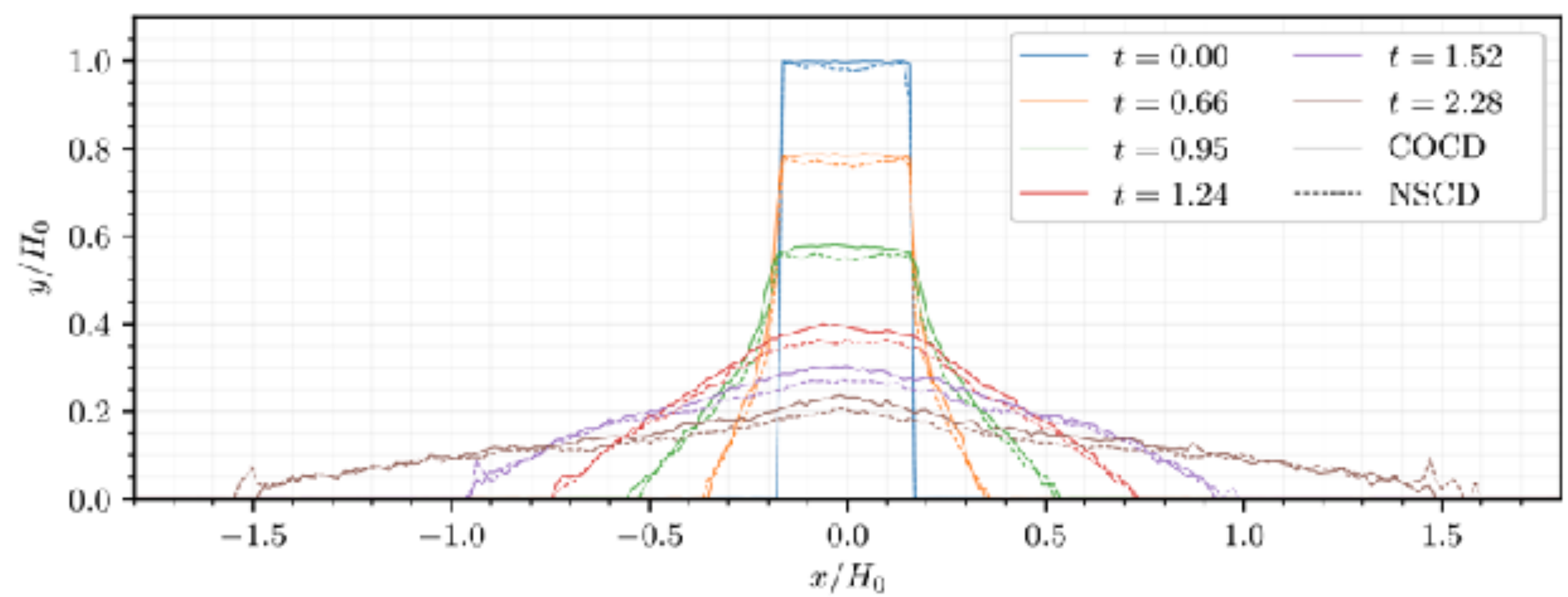
Numerical study:



$$\text{err} = O(dt)$$

- Influence of convexification?

Granular column collapse:



- An optimization-based model for dry granular flows: application to granular collapse on erodible beds, with H. Martin, A. Mangeney, Y. Maday, B. Maury, Submitted, hal-03790427
- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

Convex optimization for friction?

- Fully implicit algorithm - the corresponding dual problem

$$\min_{|\mathbf{f}_t| \leq \mu f_n} f_n \left[D^n + \Delta t\, \mathbf{n}^n \cdot A^n \left(\frac{\mathbf{u_f} + \bar{\mathbf{u}}^{n+1}}{2} \right) \right]$$

$$+ \left\langle \mathbf{f}_t, \Delta t\, P^n A^n \left(\frac{\mathbf{u_f} + \bar{\mathbf{u}}^{n+1}}{2} \right) \right\rangle$$

- Can be solved using Projected Gradient Algorithms

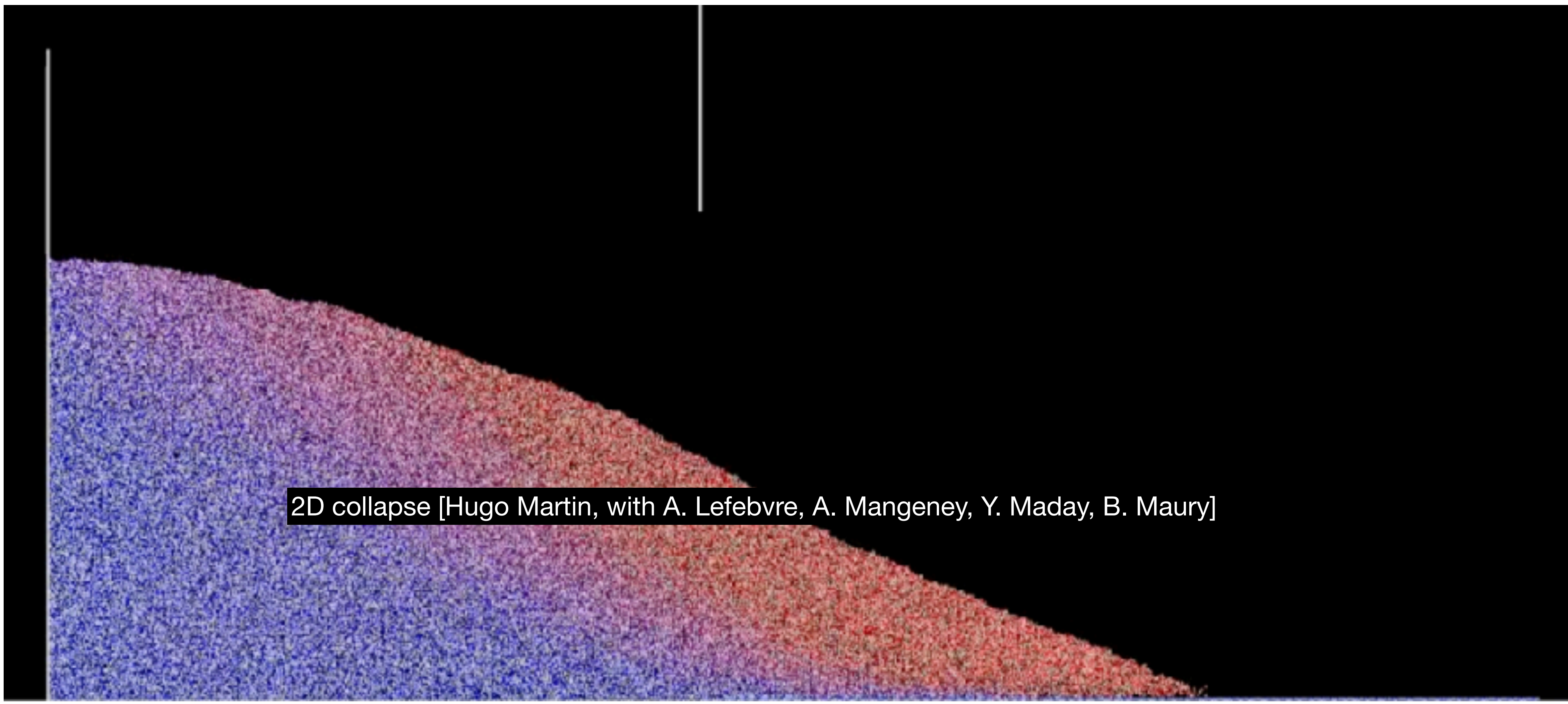
- Melanz, D., Fang, L., Jayakumar, P., & Negrut, D. (2017). A comparison of numerical methods for solving multibody dynamics problems with frictional contact modeled via differential variational inequalities. *Computer Methods in Applied Mechanics and Engineering*, *320*, 668-693.

- Corona, E., Gorsich, D., Jayakumar, P., & Veerapaneni, S. (2019). Tensor train accelerated solvers for nonsmooth rigid body dynamics. *Applied Mechanics Reviews*, *71*(5), 050804.

- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, Proceeding SMAI2021, Accepted for publication

Application: granular collapse

- Friction
- Spherical particles
- Hugo Martin [LJLL, IPGP]

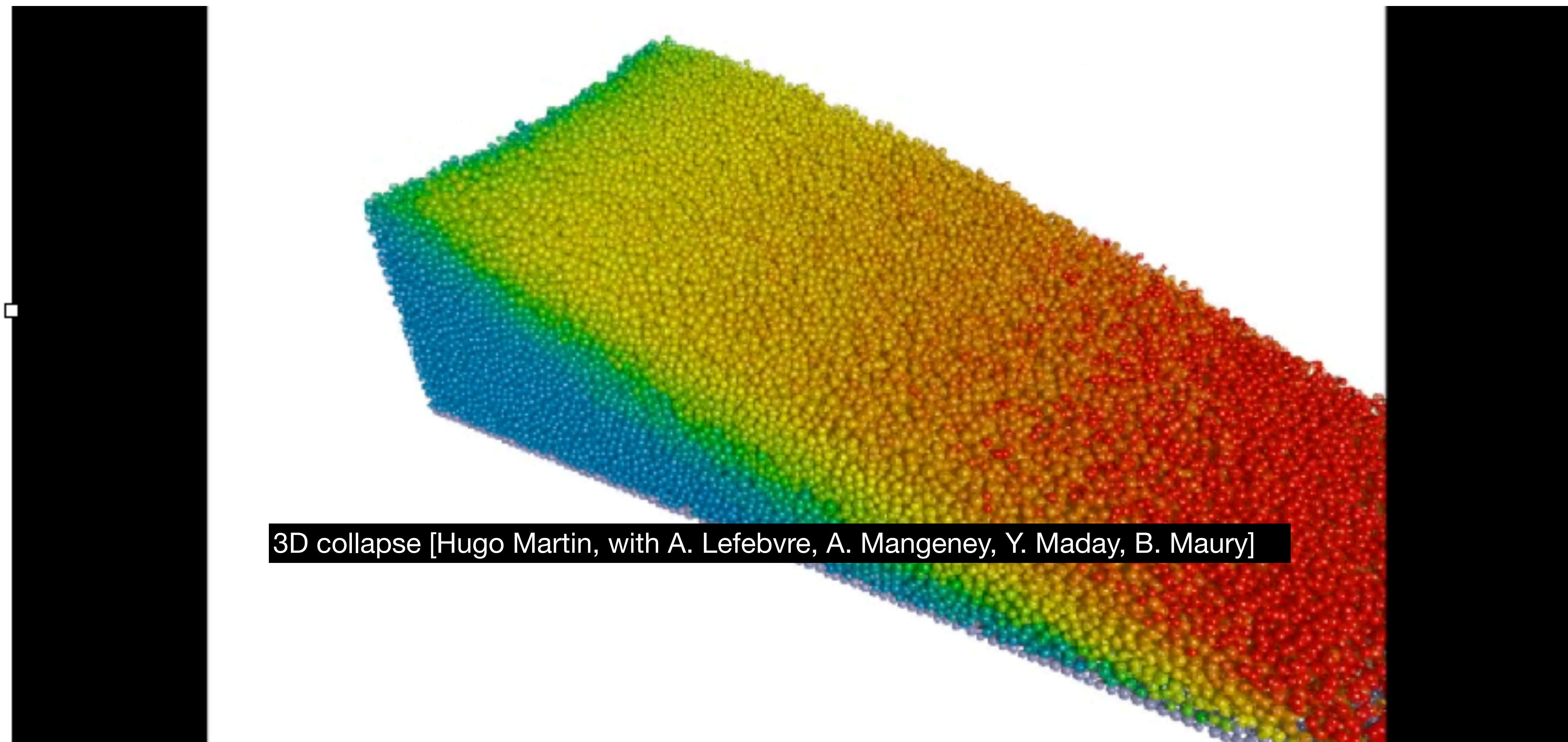


2D collapse [Hugo Martin, with A. Lefebvre, A. Mangeney, Y. Maday, B. Maury]

• An optimization-based model for dry granular flows: application to granular collapse on erodible beds, with H. Martin, A. Mangeney, Y. Maday, B. Maury, Submitted, hal-03790427

Application: granular collapse

- Friction
- Spherical particles
- Hugo Martin [LJLL, IPGP]

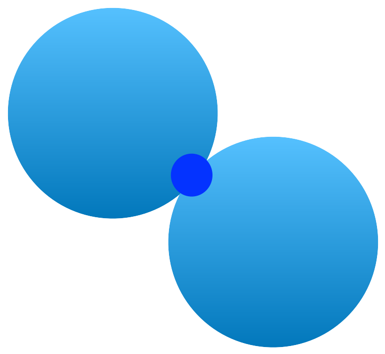


3D collapse [Hugo Martin, with A. Lefebvre, A. Mangeney, Y. Maday, B. Maury]

- An optimization-based model for dry granular flows: application to granular collapse on erodible beds, with H. Martin, A. Mangeney, Y. Maday, B. Maury, Submitted, hal-03790427

Contact Dynamics Model including lubrication in suspensions

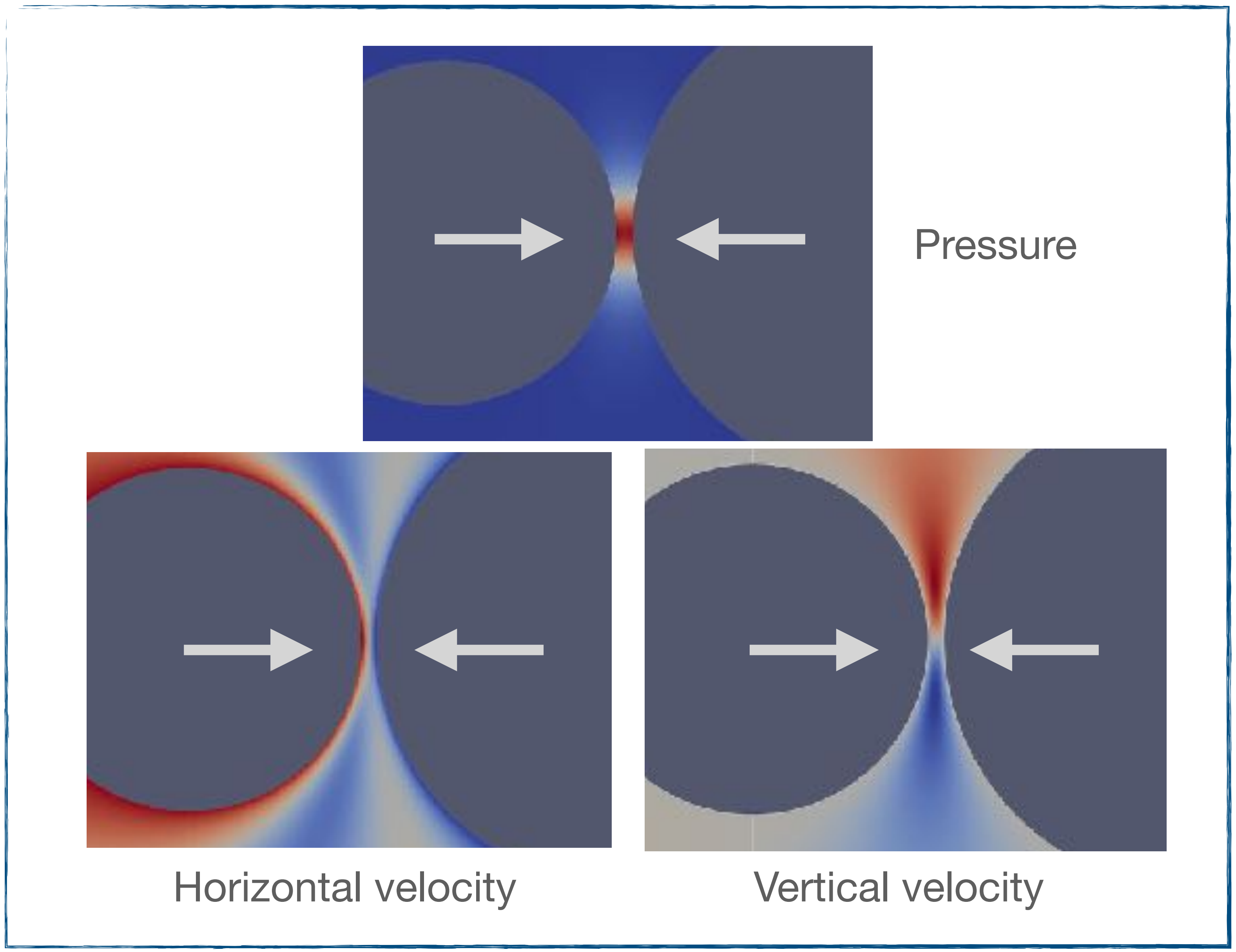
Contact law

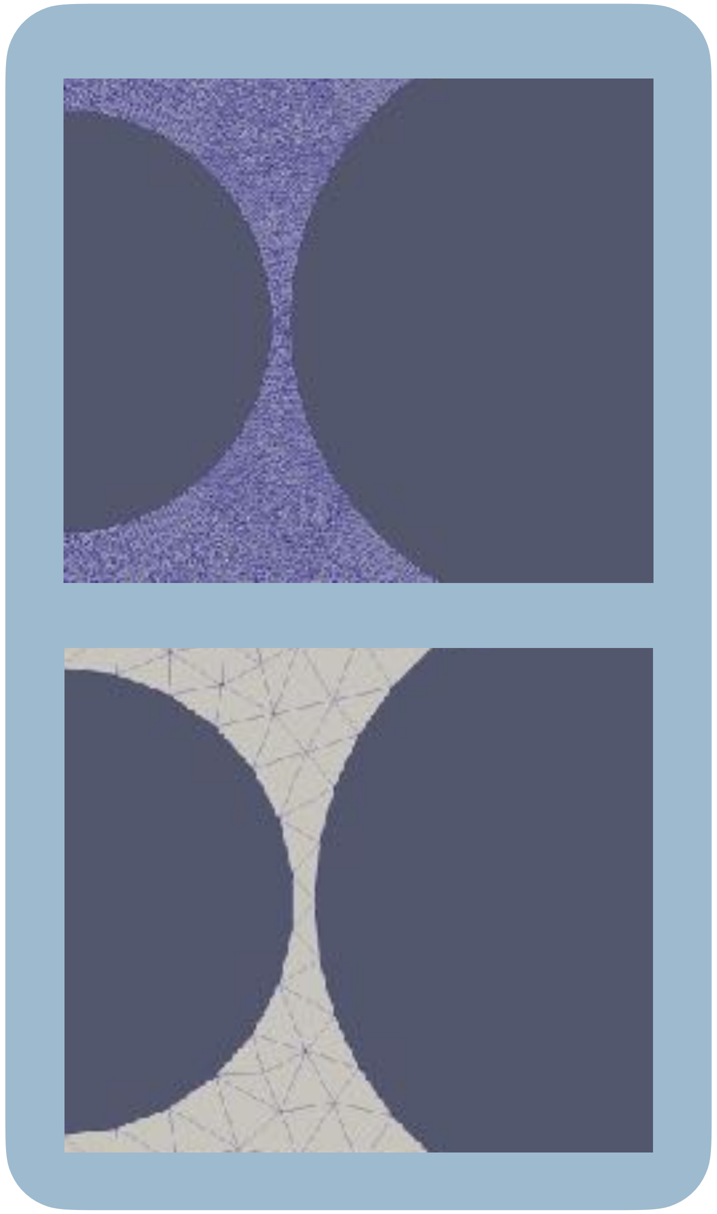


Gluey contact model

Lubrication and Numerical simulations.

- Dense suspensions $\Rightarrow$ Close interactions due to the fluid

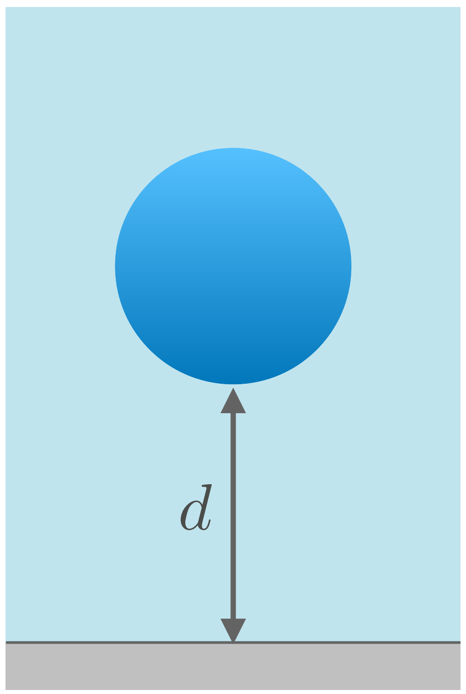




Need for methods taking lubrication into account

A stiff problem.

Lubrication force



$$F_{lub}(d) = -6\pi\mu r^2 \frac{\dot{d}}{d}$$

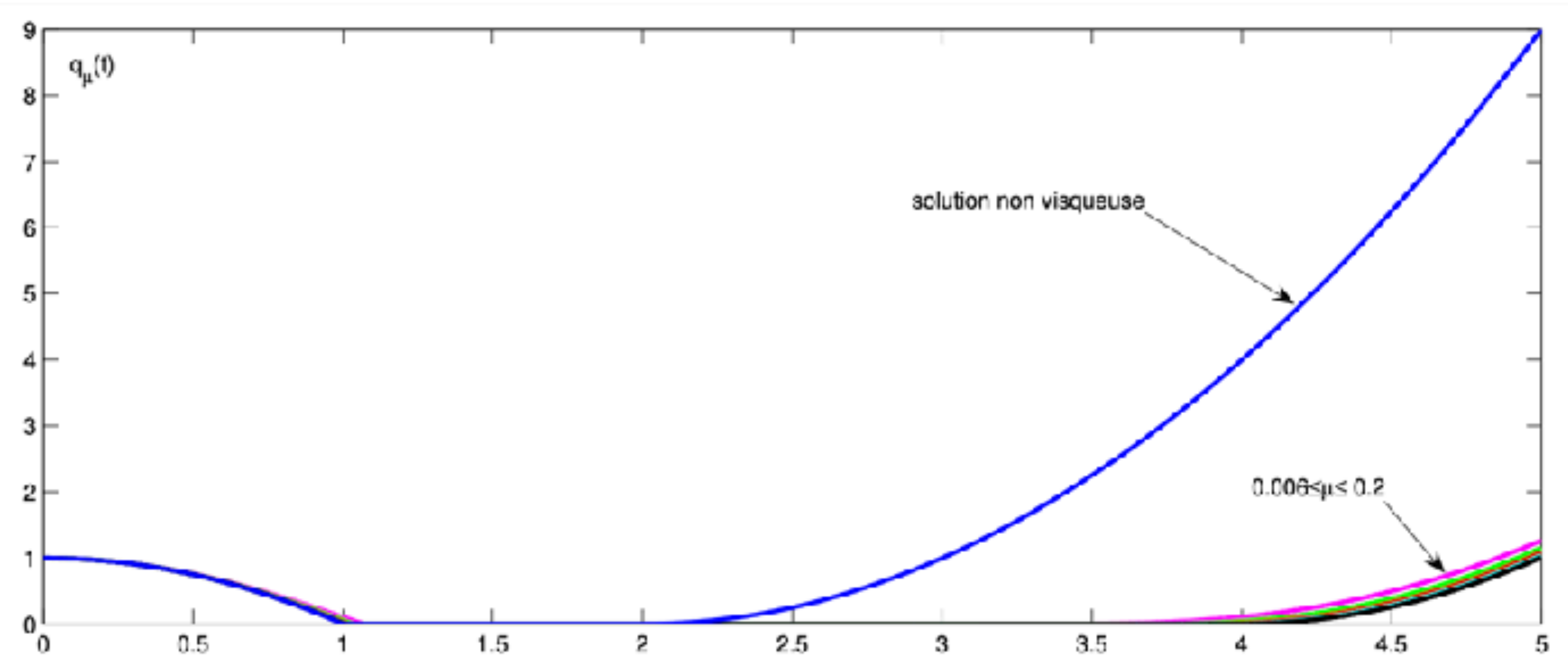
[Cox, 1974]

$$m\ddot{q} = -6\pi\mu r^2 \frac{\dot{q}}{q} + m f_y$$

- $q(t) > 0$ for any t
- BUT very small…

- Cox, R. G. (1974). The motion of suspended particles almost in contact. *International Journal of Multiphase Flow*, *1*(2), 343-371.

Modeling lubrication: the gluey contact model.

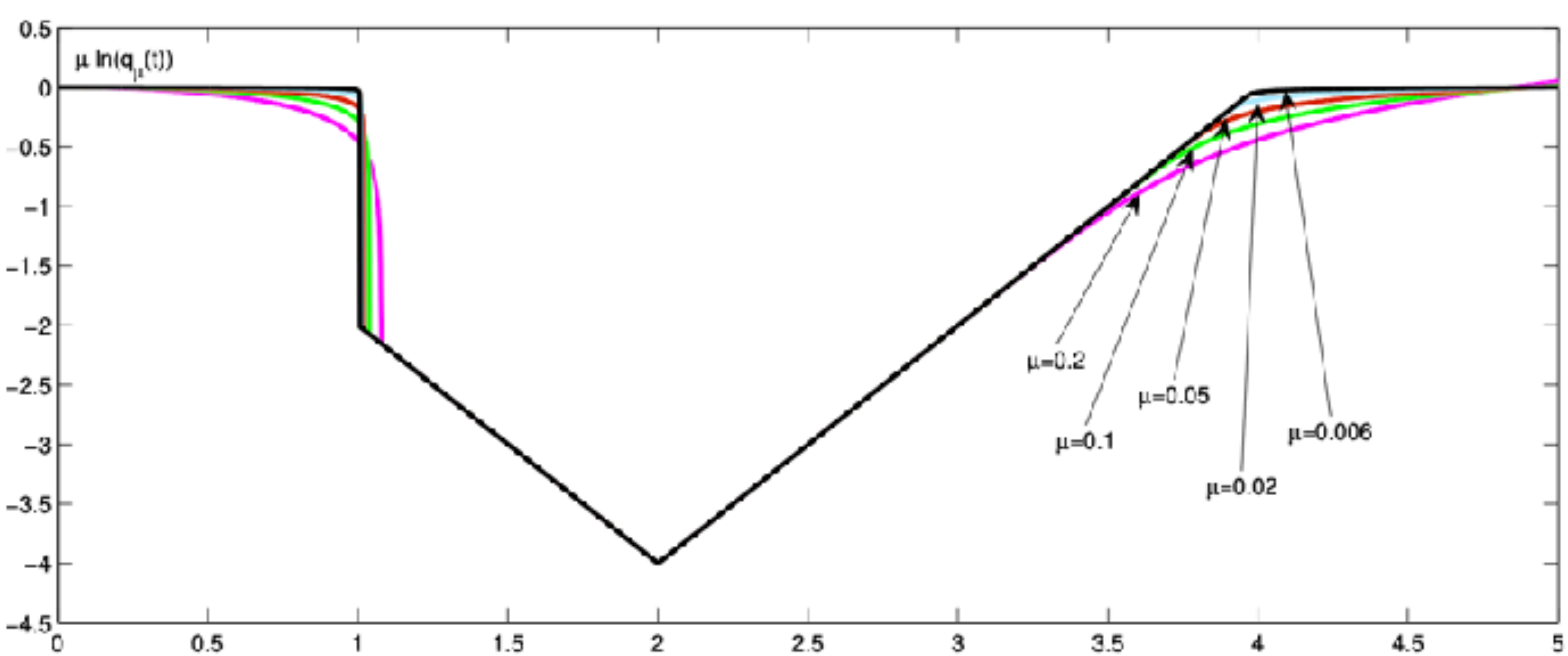


$$f_y = -2\,\mathbf{1}_{[0,2]} + 2\,\mathbf{1}_{[2,+\infty]}$$

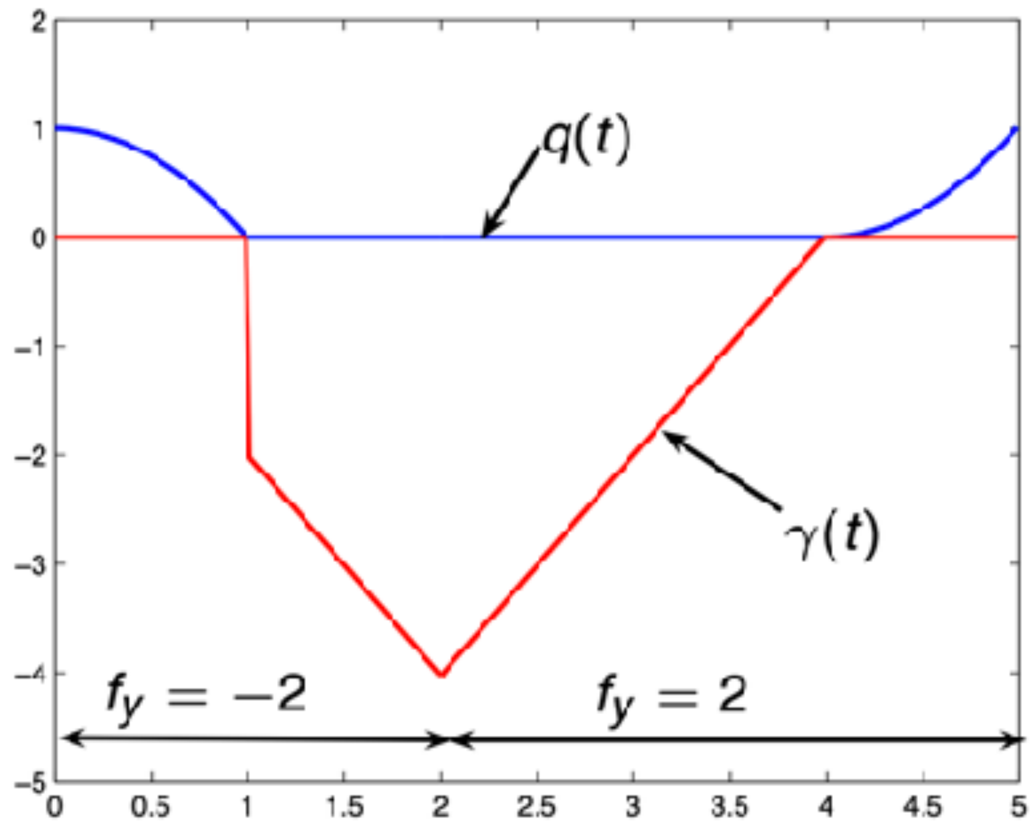
$$\ddot{q}_\mu = -\mu \frac{\dot{q}_\mu}{q_\mu} + f_y$$

$$\dot{q}_\mu(0) = 0$$

$$q_\mu(0) = 1$$



Modeling lubrication: the gluey contact model.



$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = m f_y + \lambda$$

$$\mathrm{supp}(\lambda) \subset \{t, q(t) = 0\}$$

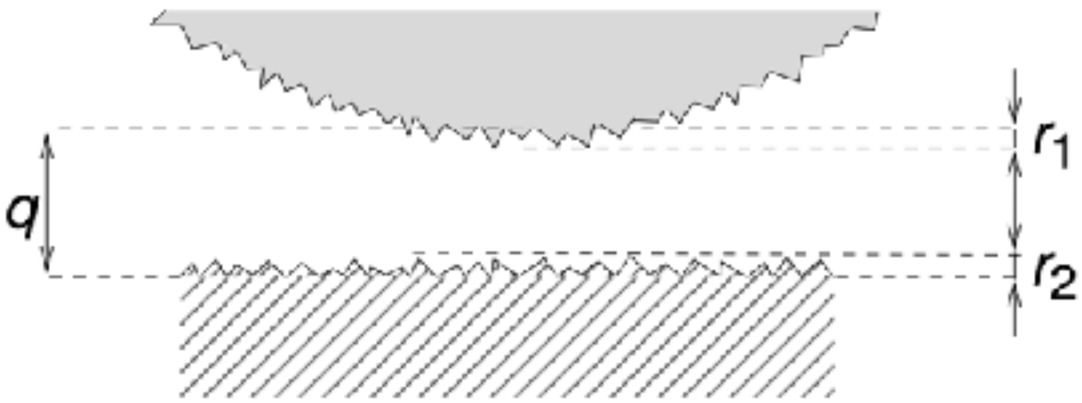
$$\dot{\gamma} = -\lambda$$

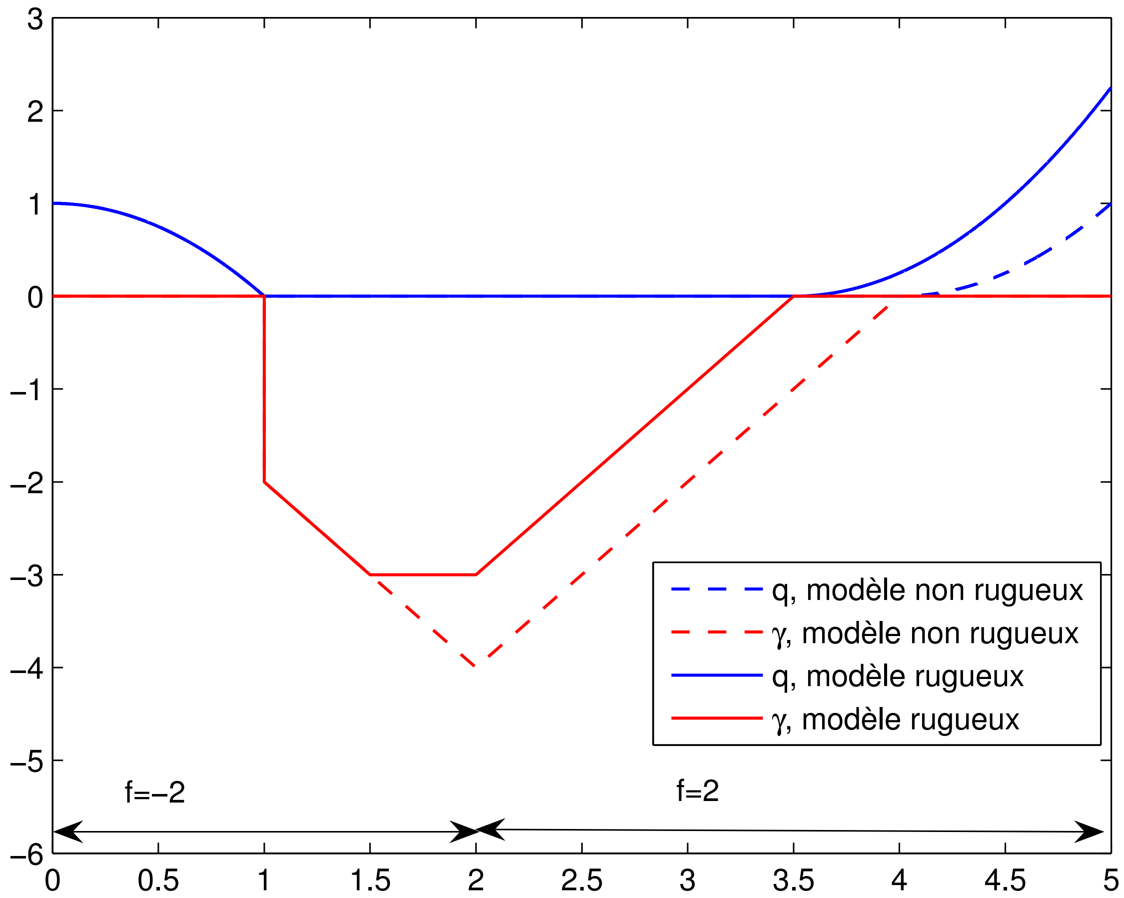
$$q \geq 0, \gamma \leq 0$$

$$C_{q,\gamma} = \left| \begin{array}{lll} \{0\} & \text{si} & \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} & \gamma^- = 0 \\ & & q = 0 \\ \mathbb{R} & \text{sinon} & \end{array} \right.$$

- B. Maury, A gluey particle model, ESAIM Proceedings, July 2007, Vol.18, 133-142
- A. Lefebvre, Numerical simulation of gluey particles, M2AN, 43:53-80 (2009)

Modeling lubrication: the gluey contact model.

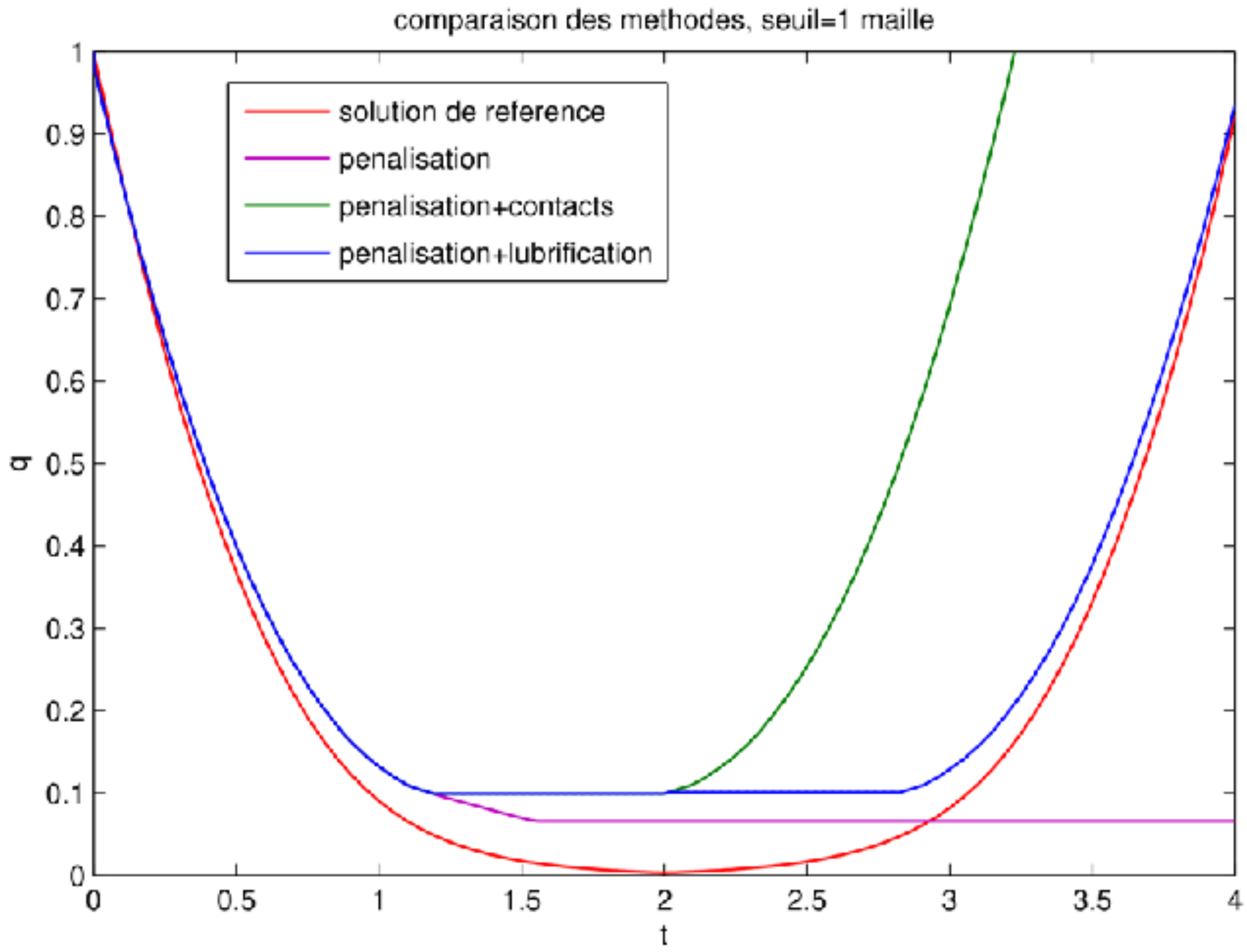


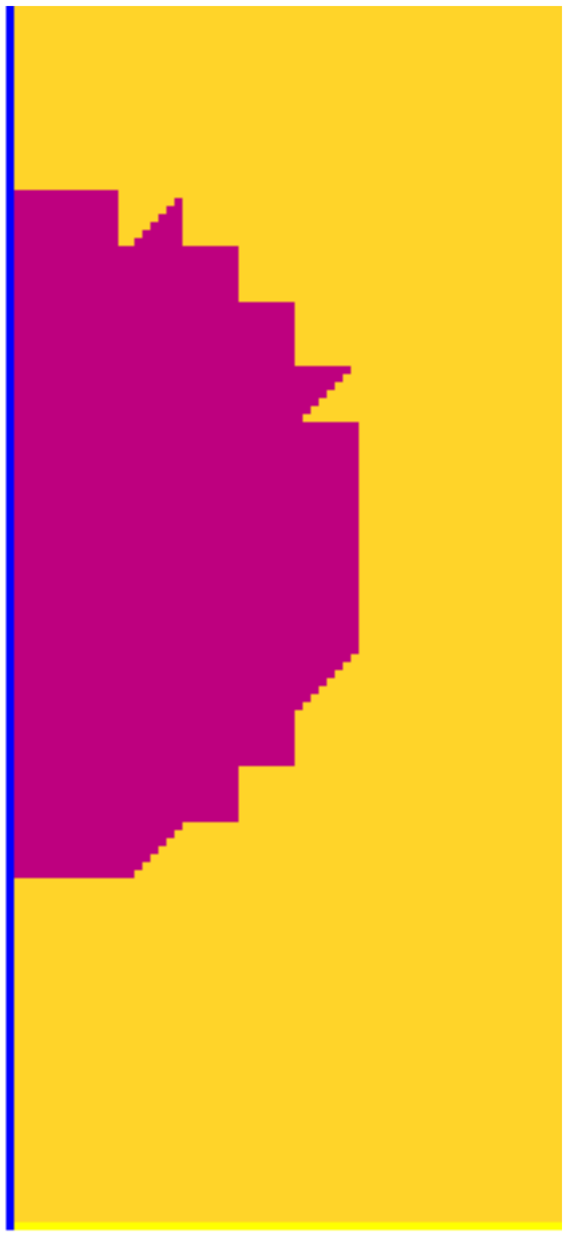


- Roughness $\Longrightarrow$ contact
- Model : contact for $q = r_1 + r_2$

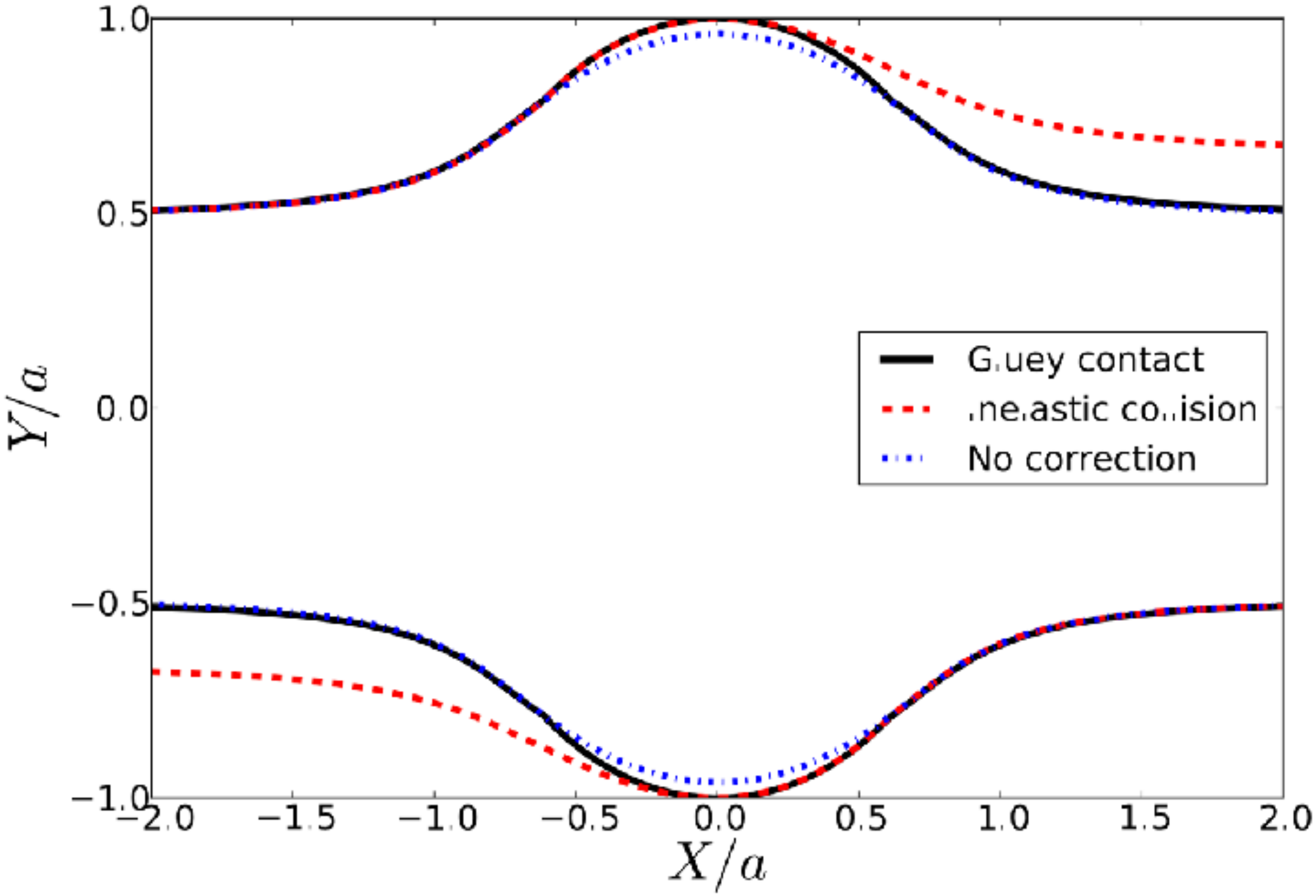
$\Longrightarrow$ Threshold for γ : $\gamma \geq \gamma_{min} = \mu \ln(r_1 + r_2)$

Coupling lubrication model and fluid solver

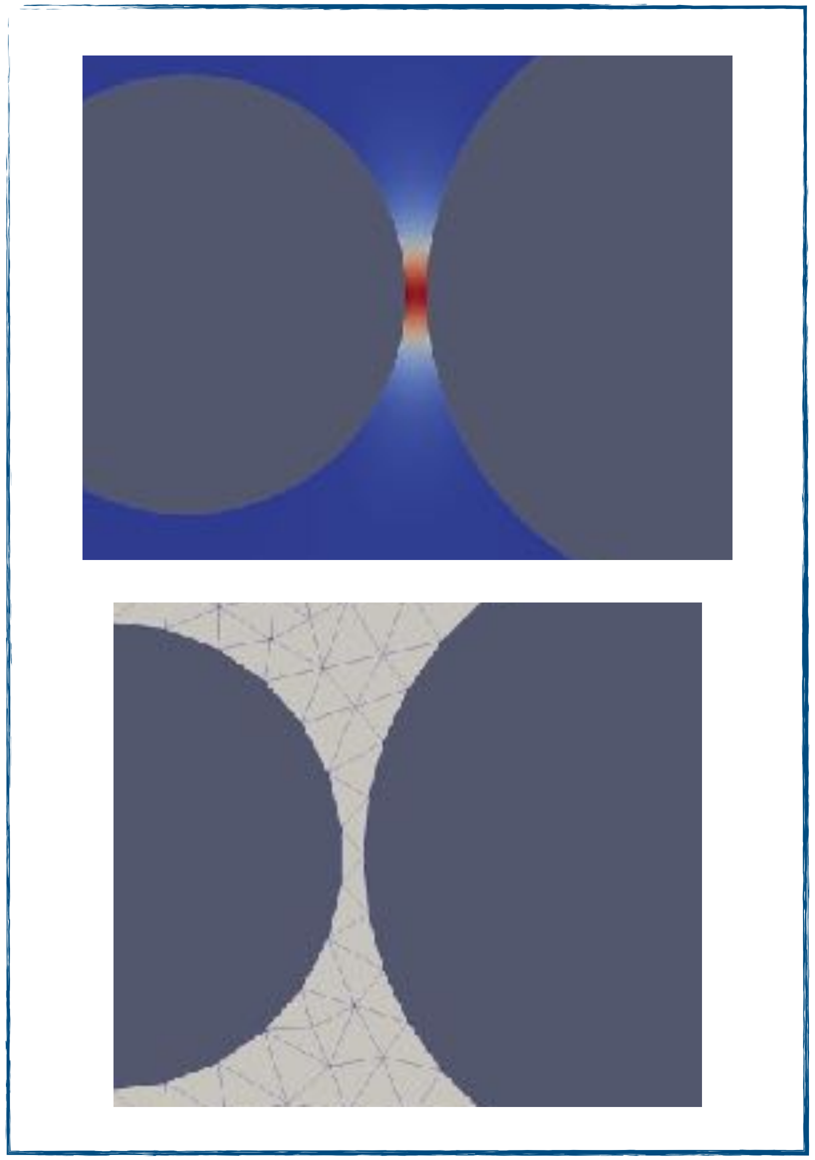




Coupling lubrication model and fluid solver

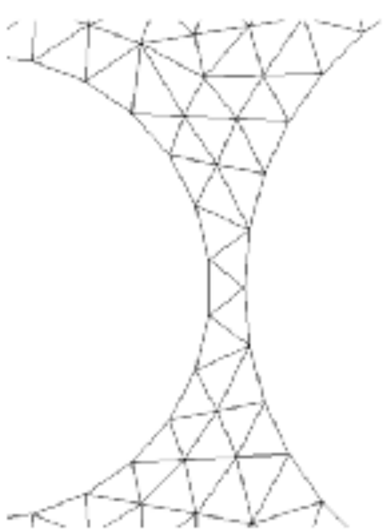


Taking lubrication into account when using direct fluid/particle solvers

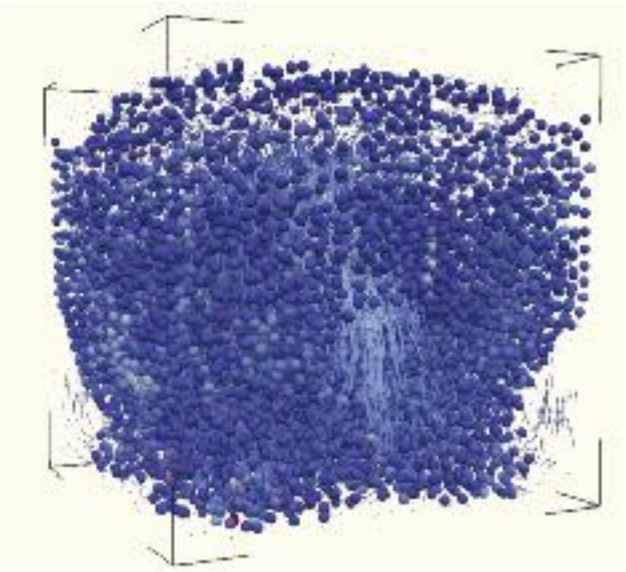


Direct numerical simulations

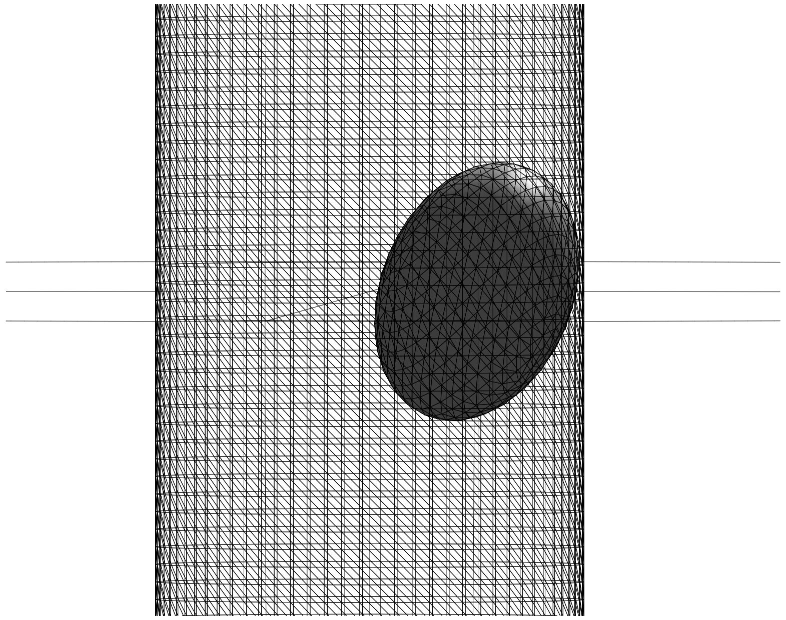
- Solving the fluid-particles PDE model
- Approximation of the velocity and pressure fields
- Underlying mesh
- Converging results



Freefem++

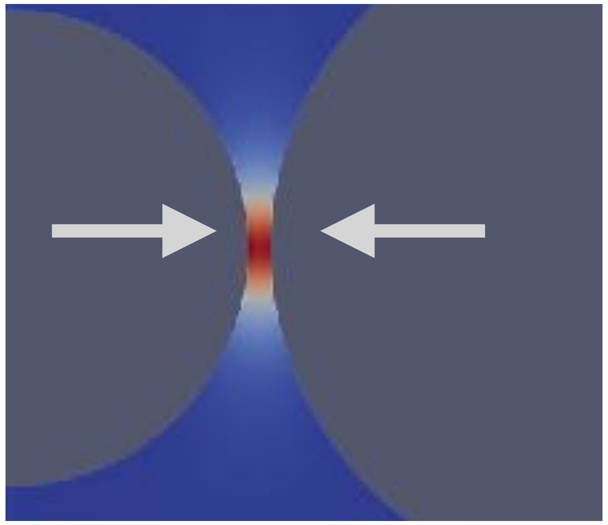


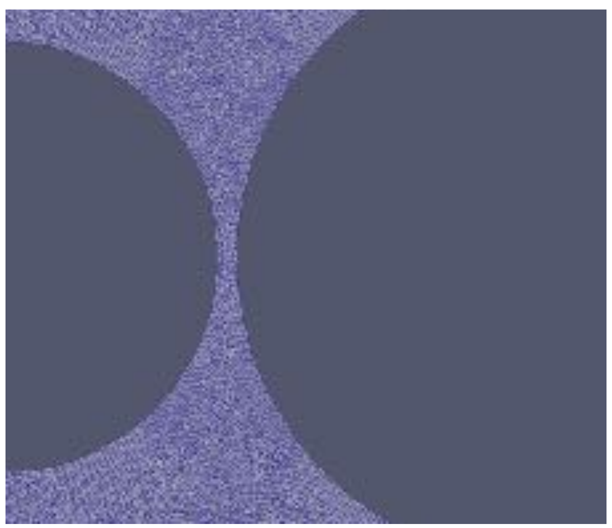
CAFES (CMAP-LMO)

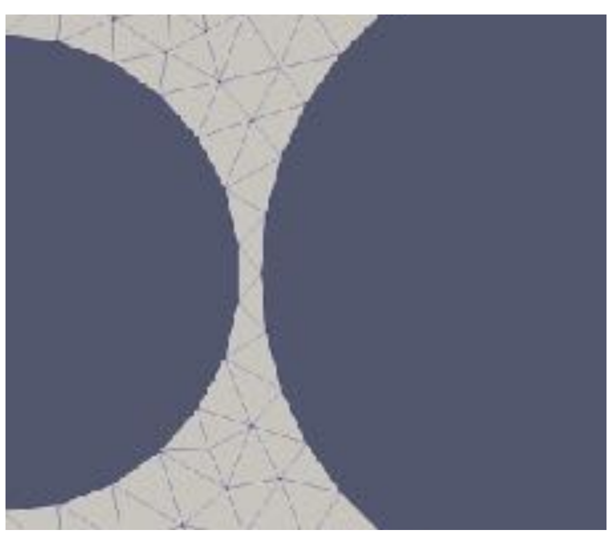


MyBEM (CMAP)

Lubrication phenomenon



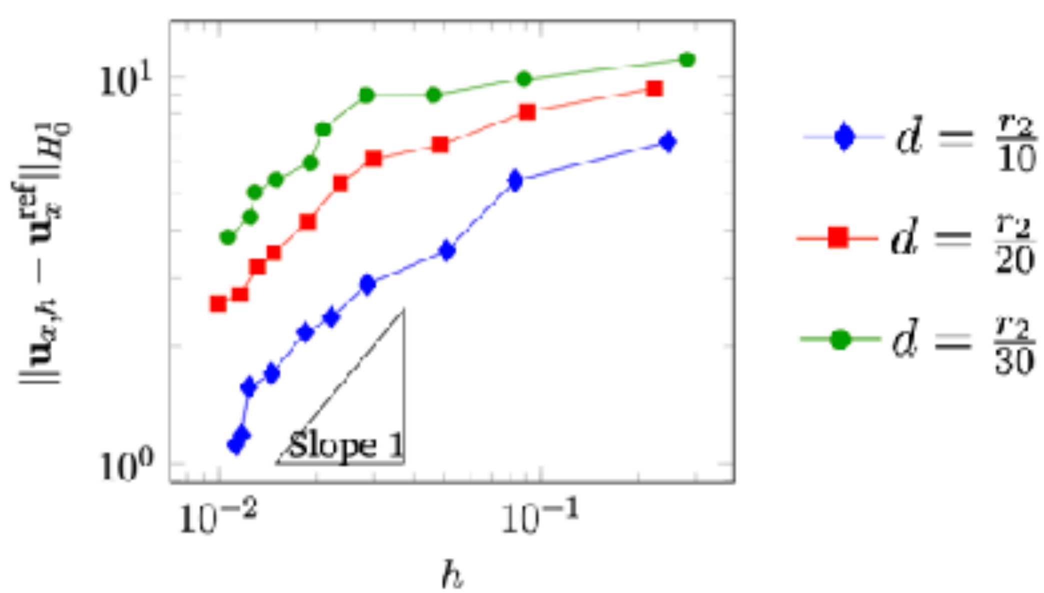




- Numerical resolution of a singular problem…

$$
\begin{aligned}
-\mu \triangle \mathbf{u} + \nabla \mathrm{p} &= 0 \quad \text{in } \mathcal{F} \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \mathcal{F} \\
\mathbf{u} &= \mathbf{u}^\star \quad \text{on } \partial B
\end{aligned}
$$

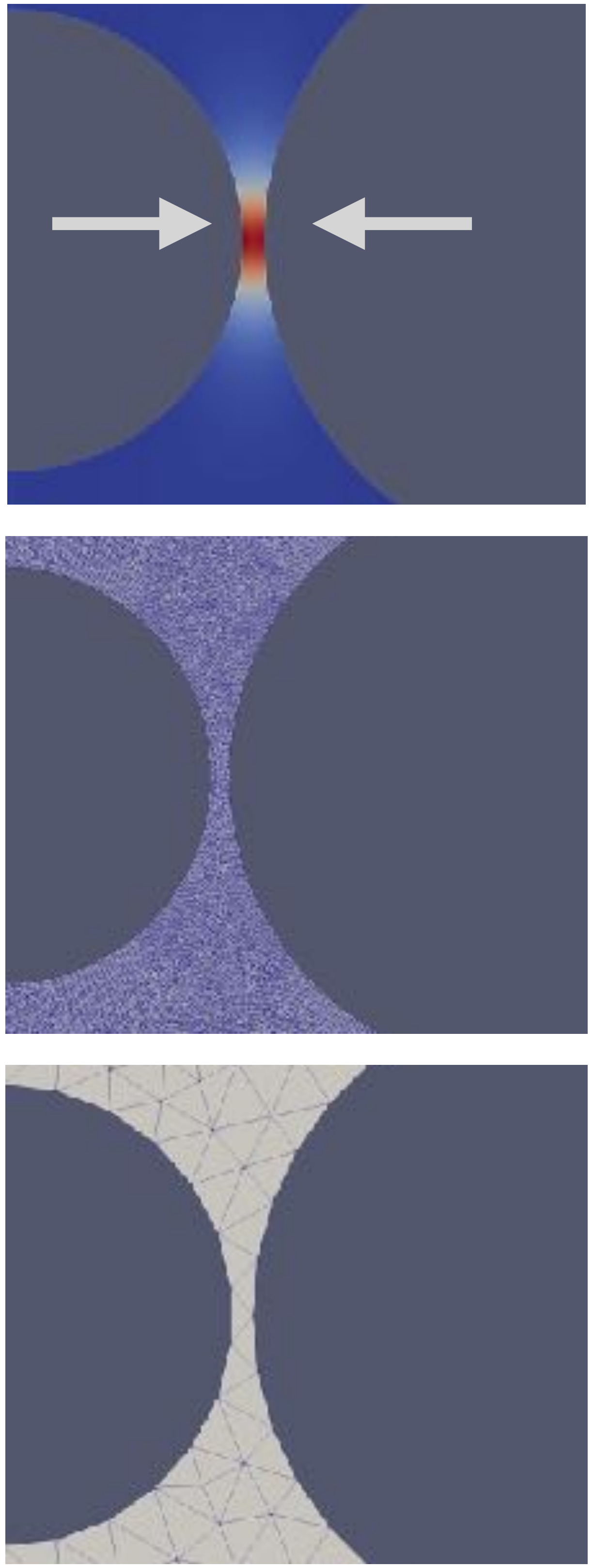
$$\|\mathbf{u} - \mathbf{u}_h\| \leq C \, \|(\mathbf{u}, \mathrm{p})\| \, h^\alpha$$



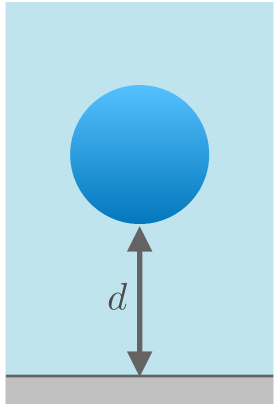
Blows up
when the distance
goes to zero…

Need for methods taking lubrication into account

An explicit asymptotic expansion



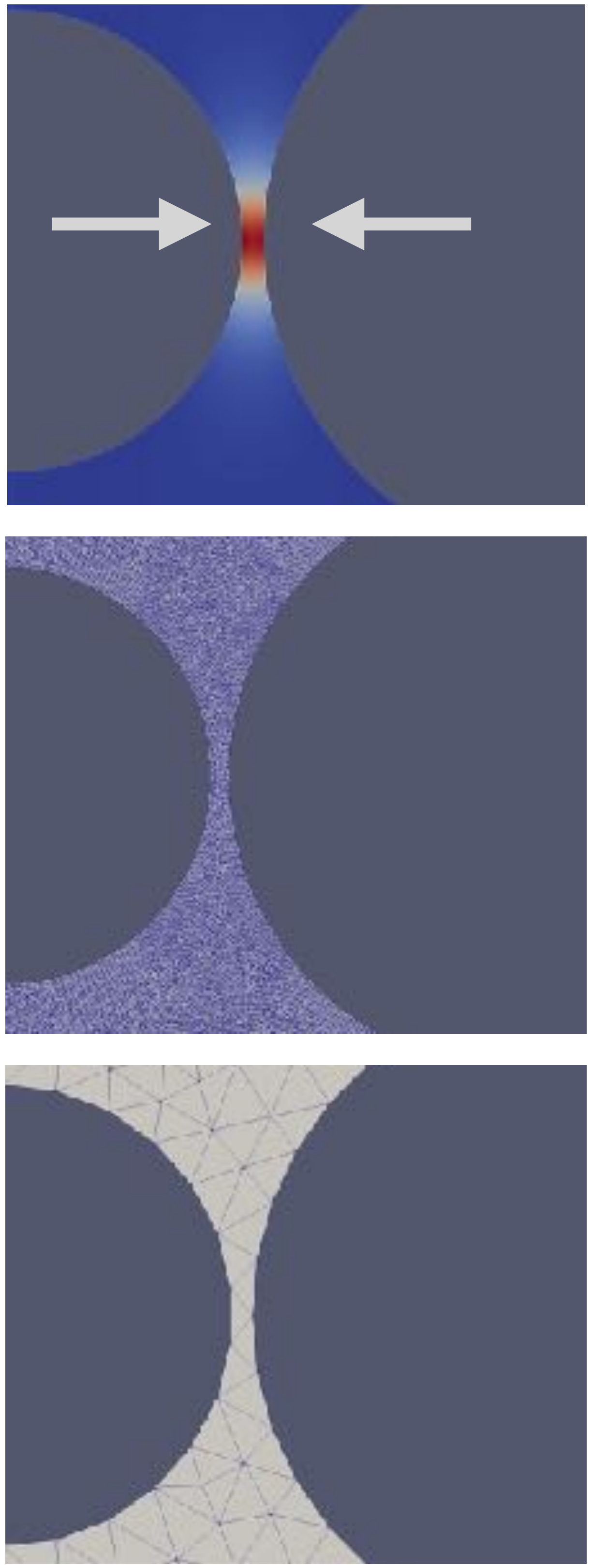
Lubrication force



$$F_{lub}(d) = -6\pi\mu r^2 \frac{\dot{d}}{d}$$

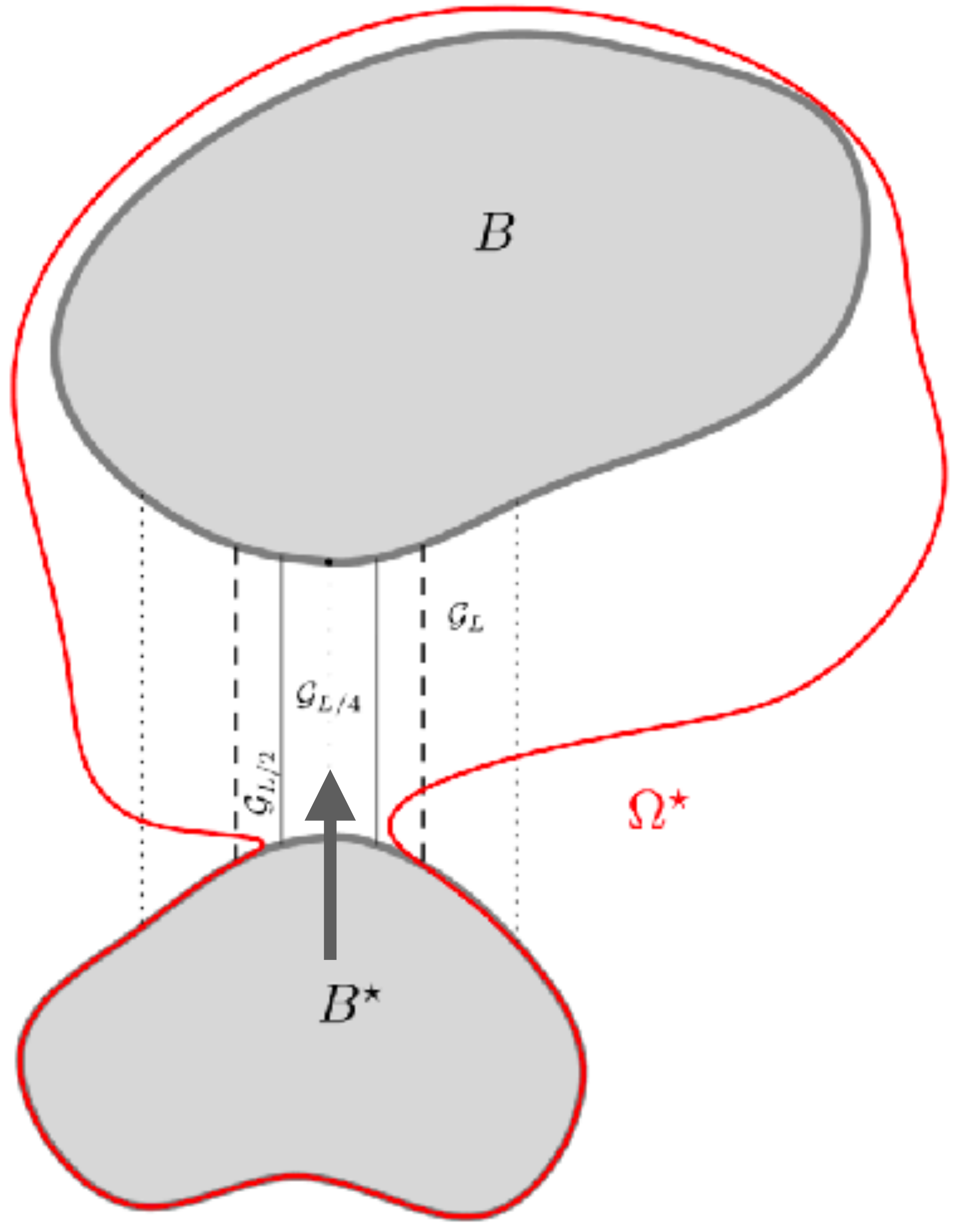
[Cox, 1974]

An explicit asymptotic expansion

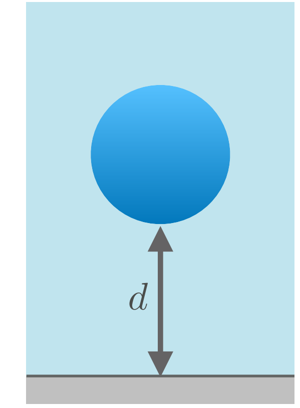


Explicit asymptotic expansion
[Hillairet, Kelai, 2015]

$$(\mathbf{u}, \mathrm{p}) = (\mathbf{u}^{\mathrm{sing}}, \mathrm{p}^{\mathrm{sing}}) + (\mathbf{u}^{\mathrm{reg}}, \mathrm{p}^{\mathrm{reg}})$$



Lubrication force



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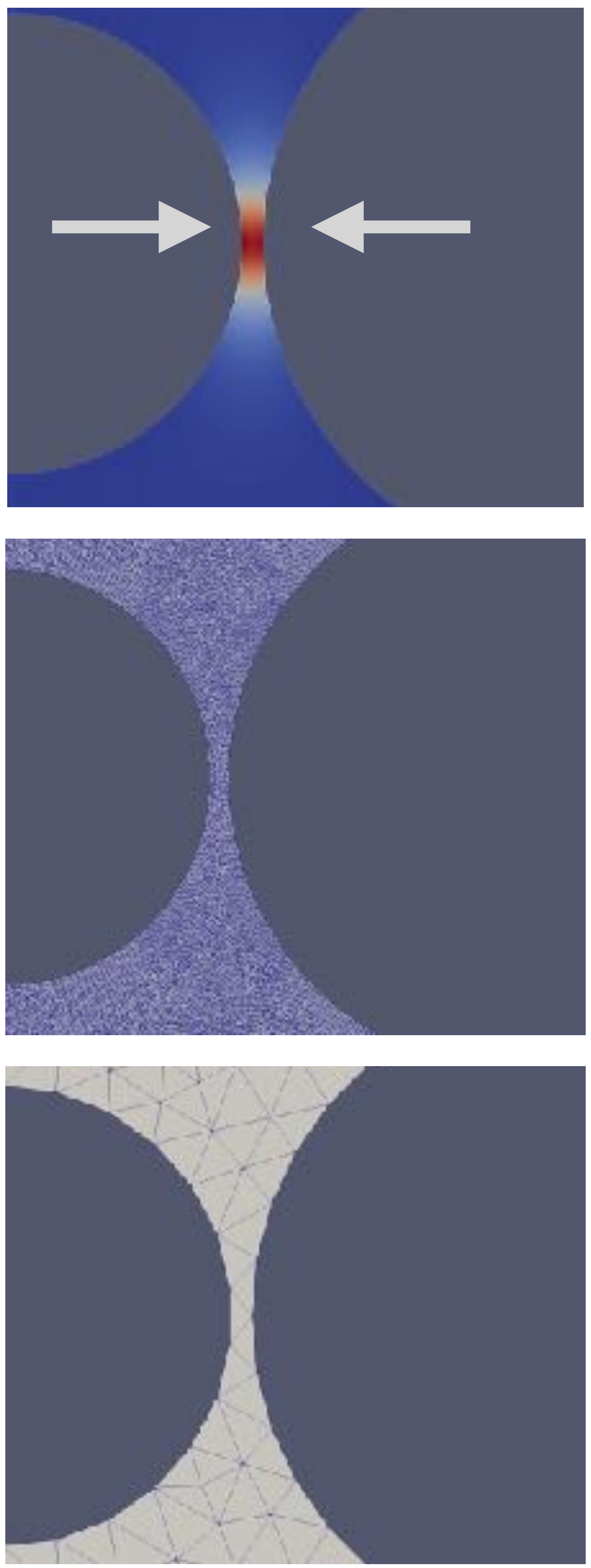

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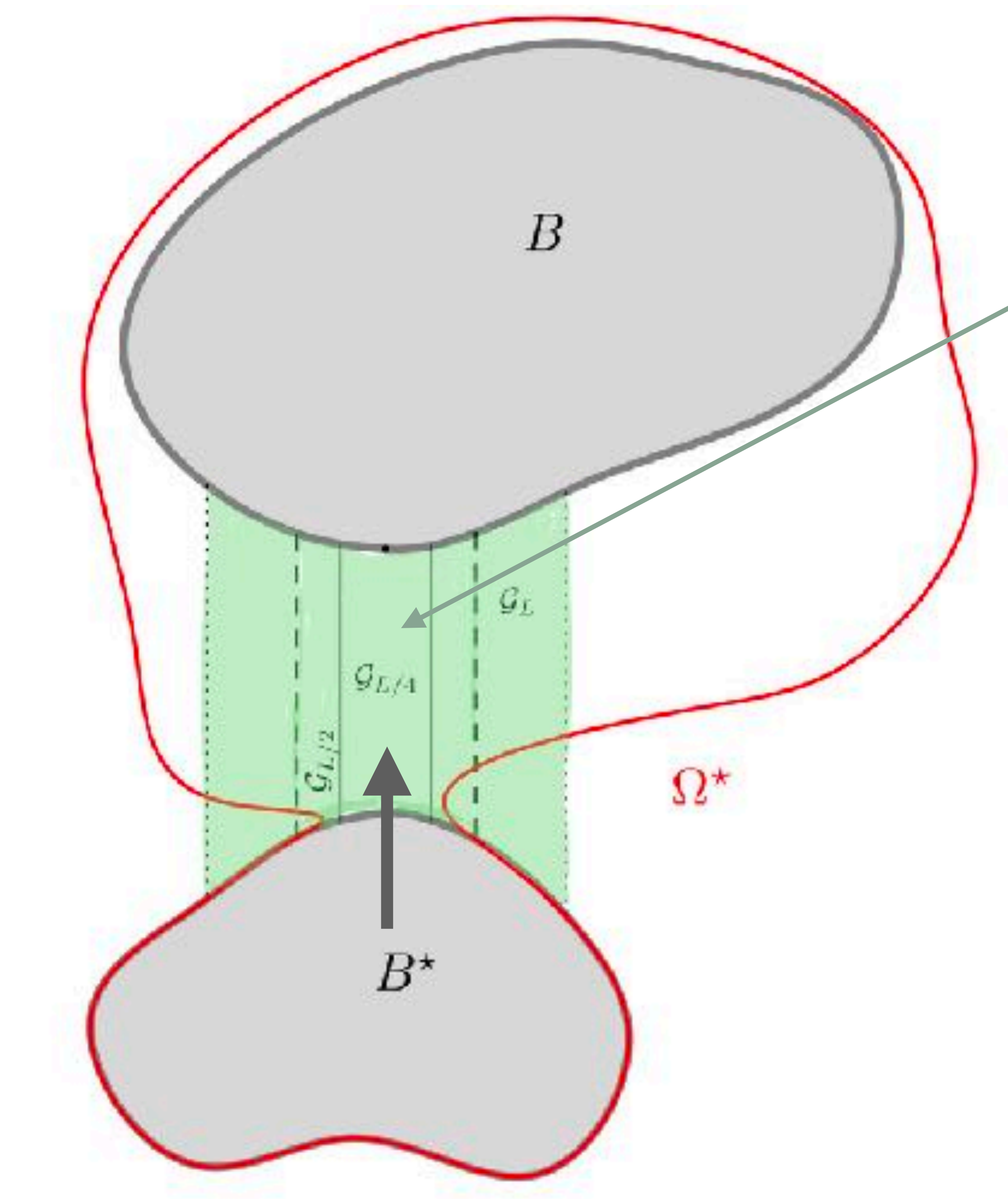
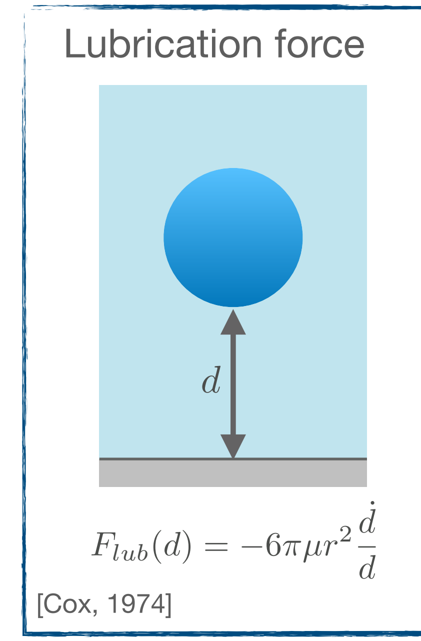
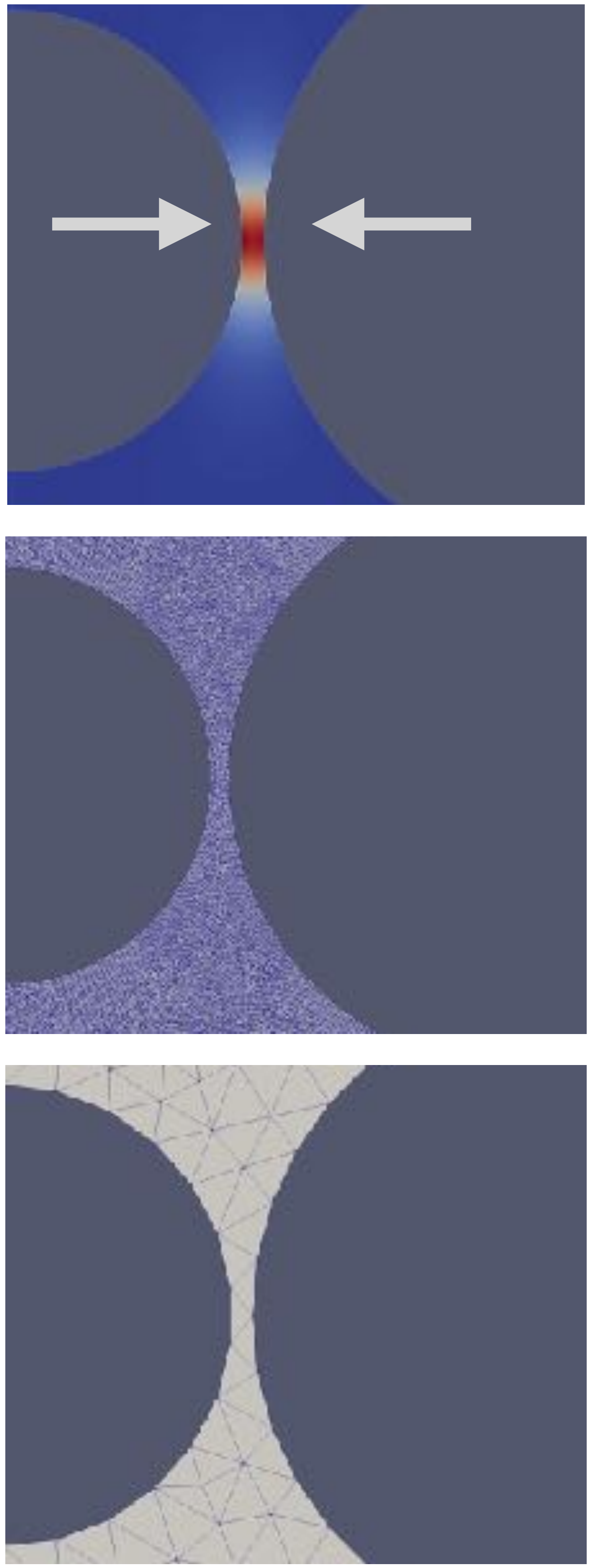


Divergence free explicit expansion



An explicit asymptotic expansion



Explicit asymptotic expansion [Hillairet, Kelai, 2015]

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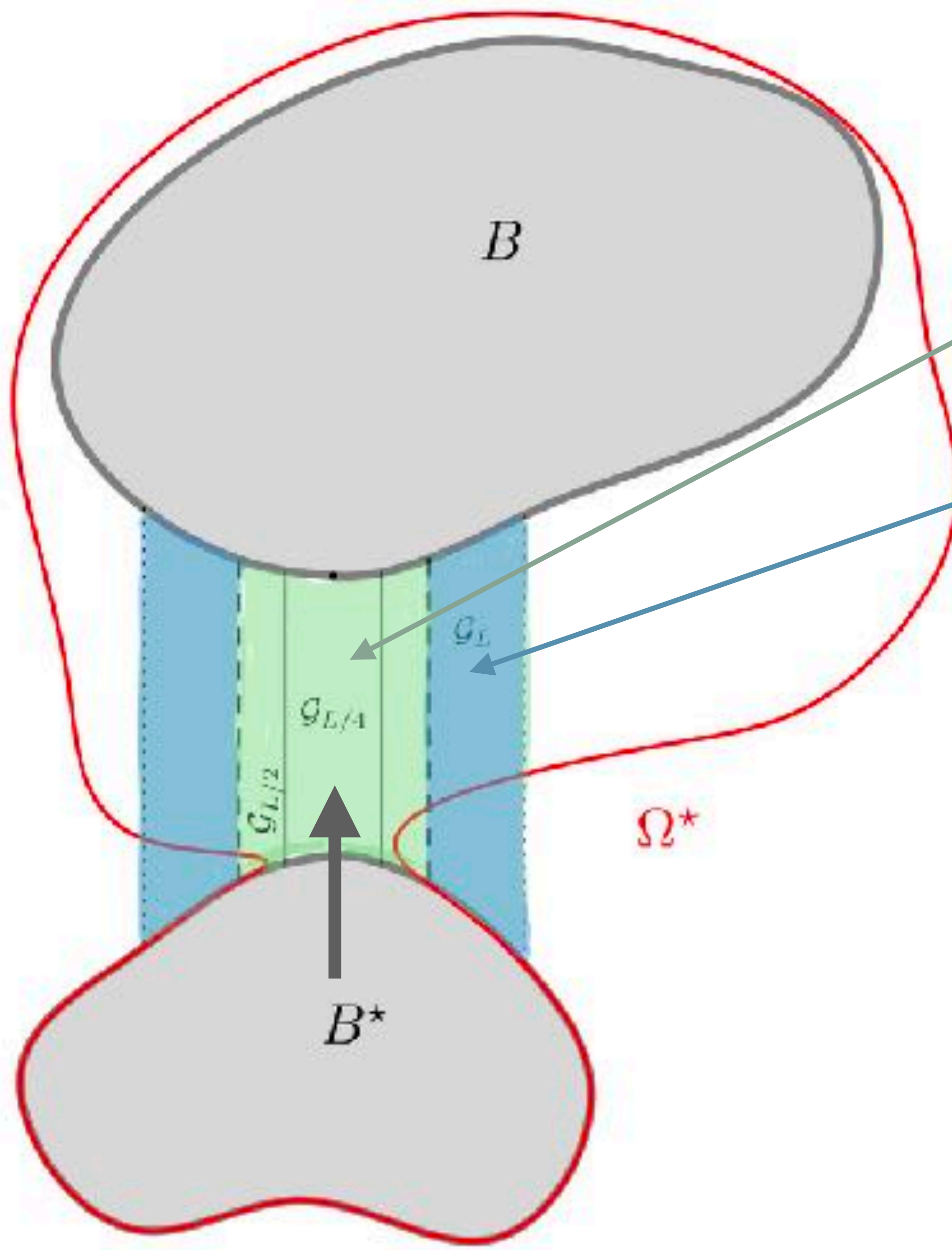
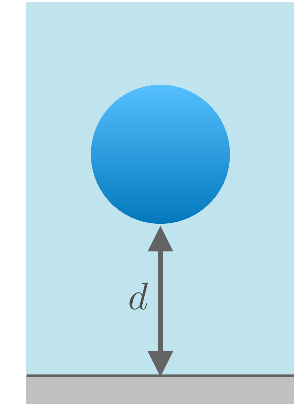


Divergence free explicit expansion

Divergence free truncation

Lubrication force



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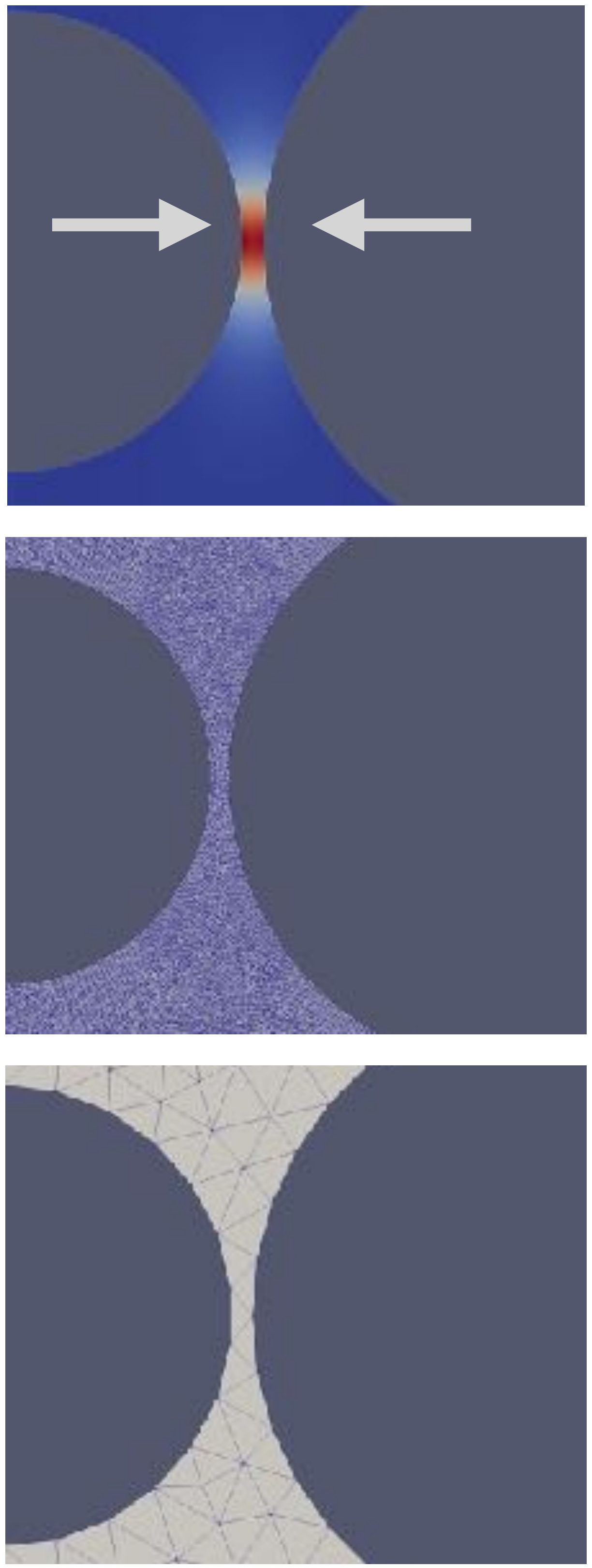
An explicit asymptotic expansion



Explicit asymptotic expansion [Hillairet, Kelai, 2015]

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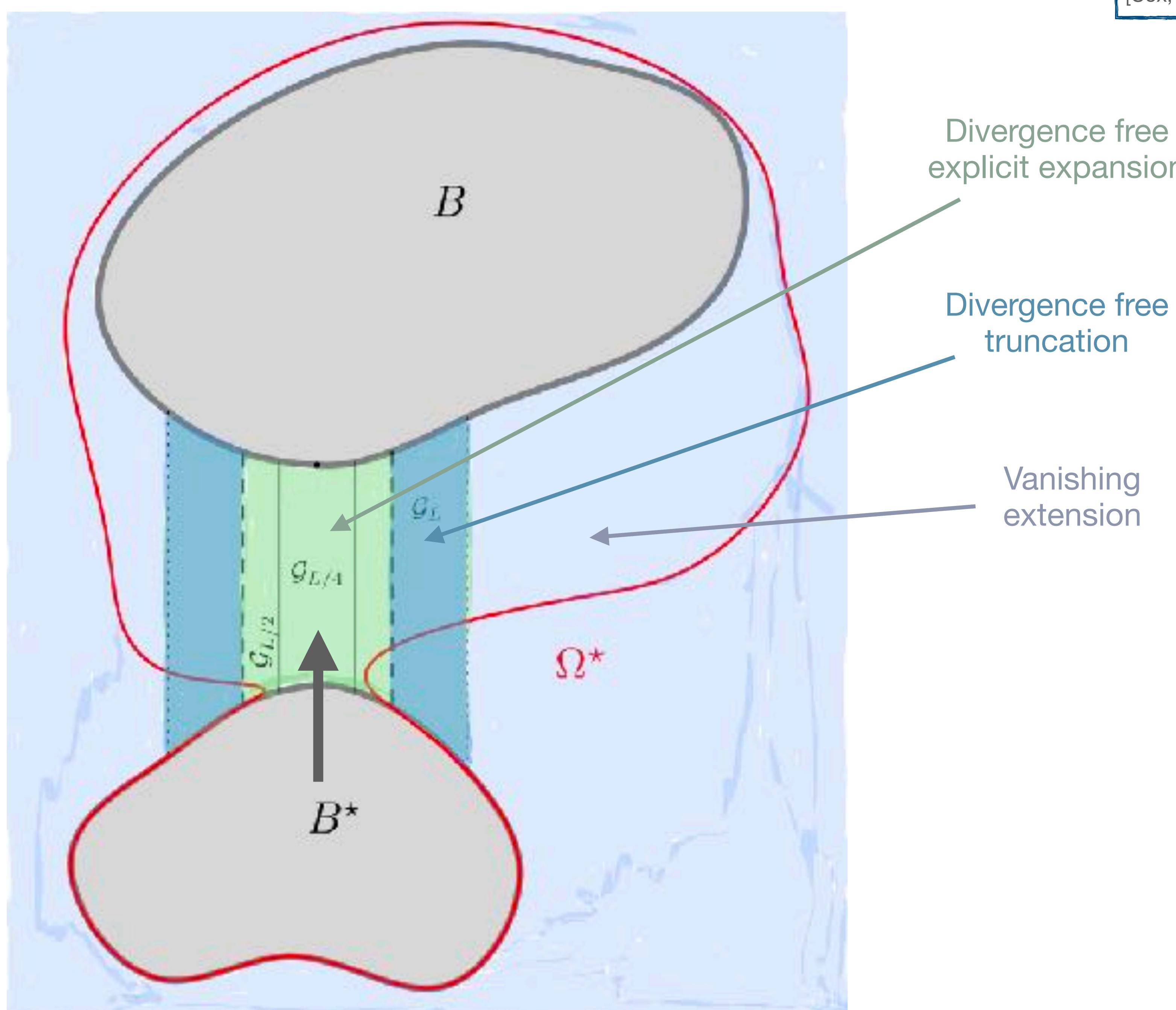

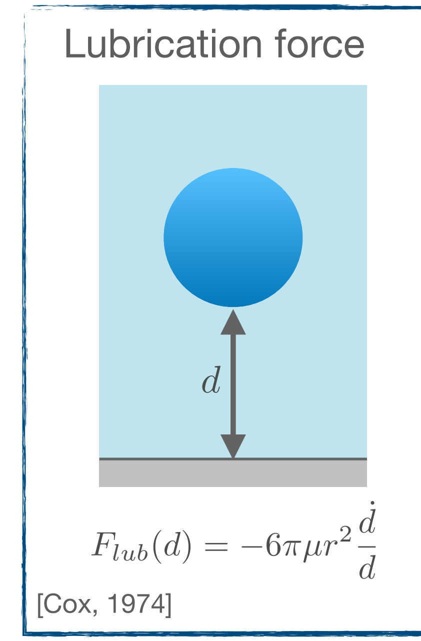

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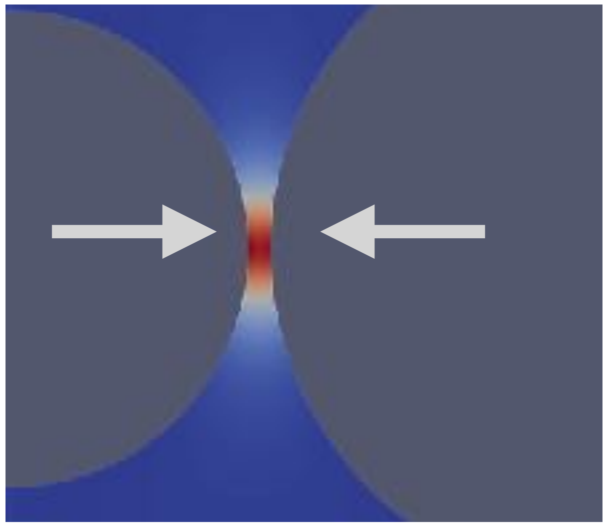

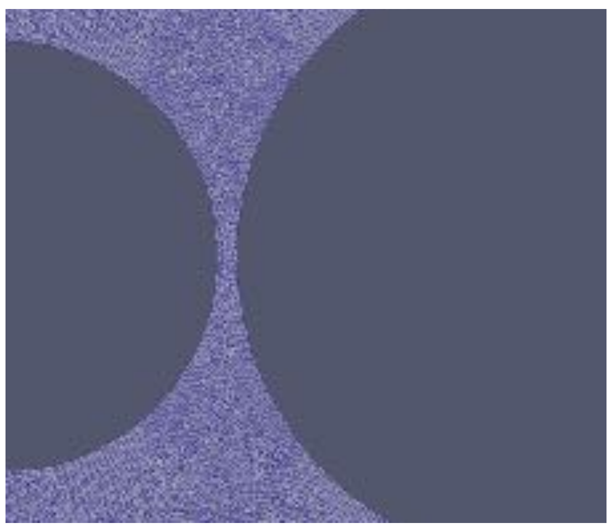

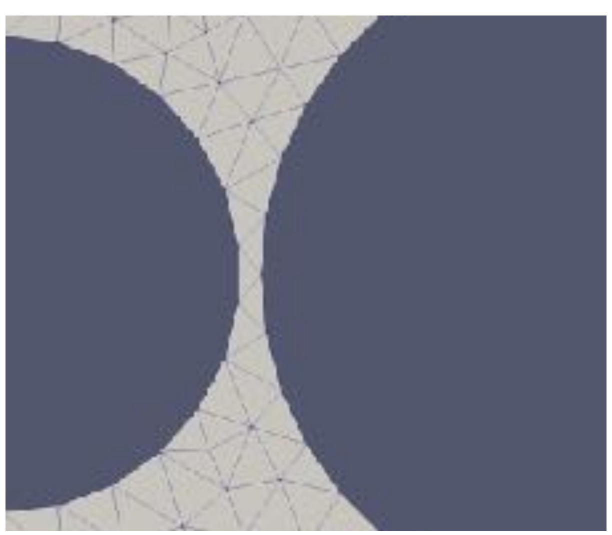

Explicit asymptotic expansion [Hillairet, Kelai, 2015]

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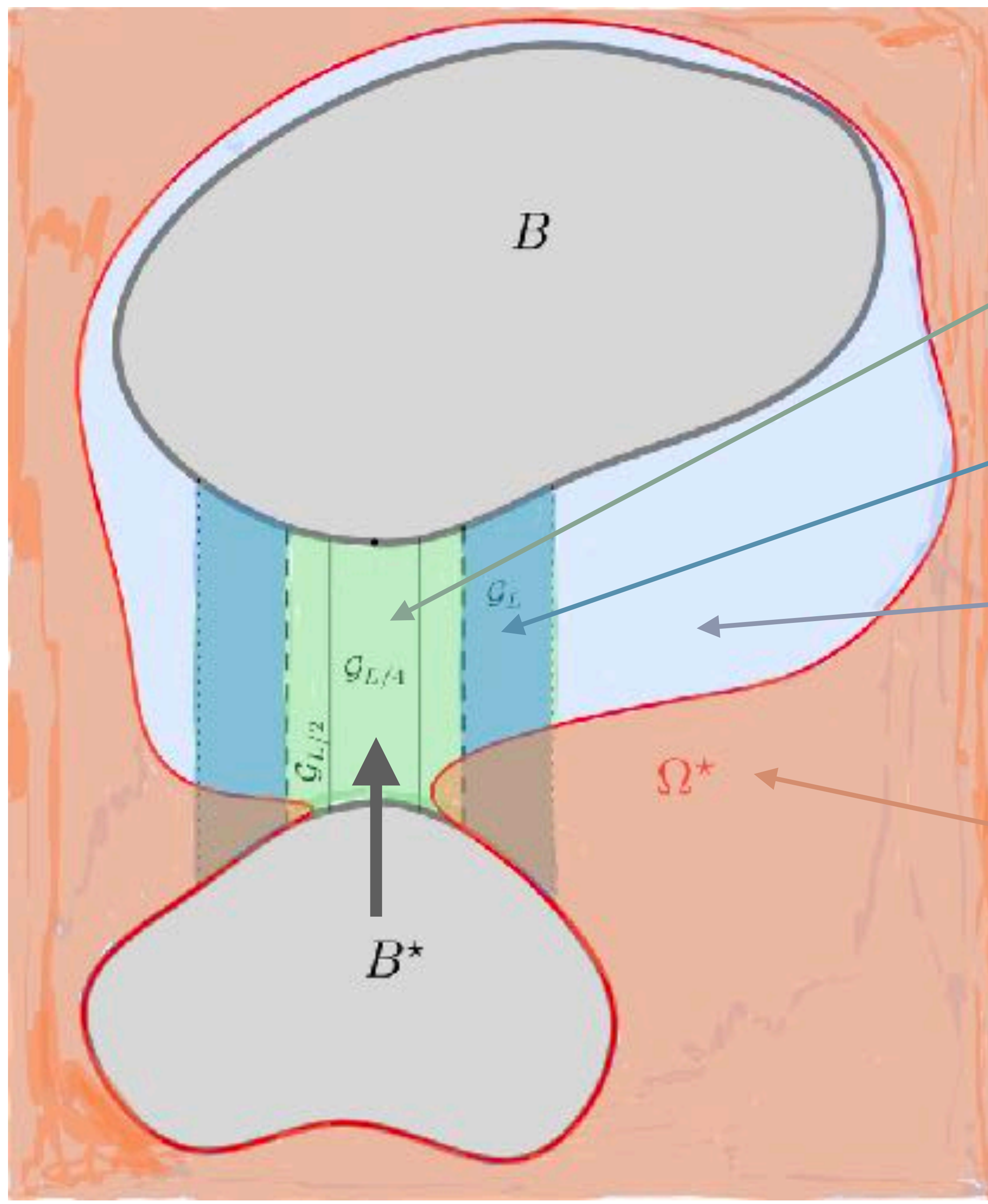
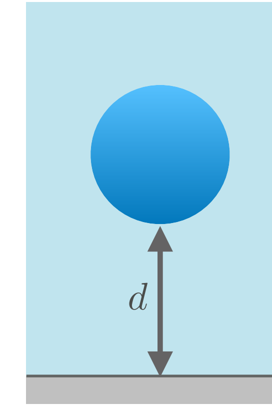


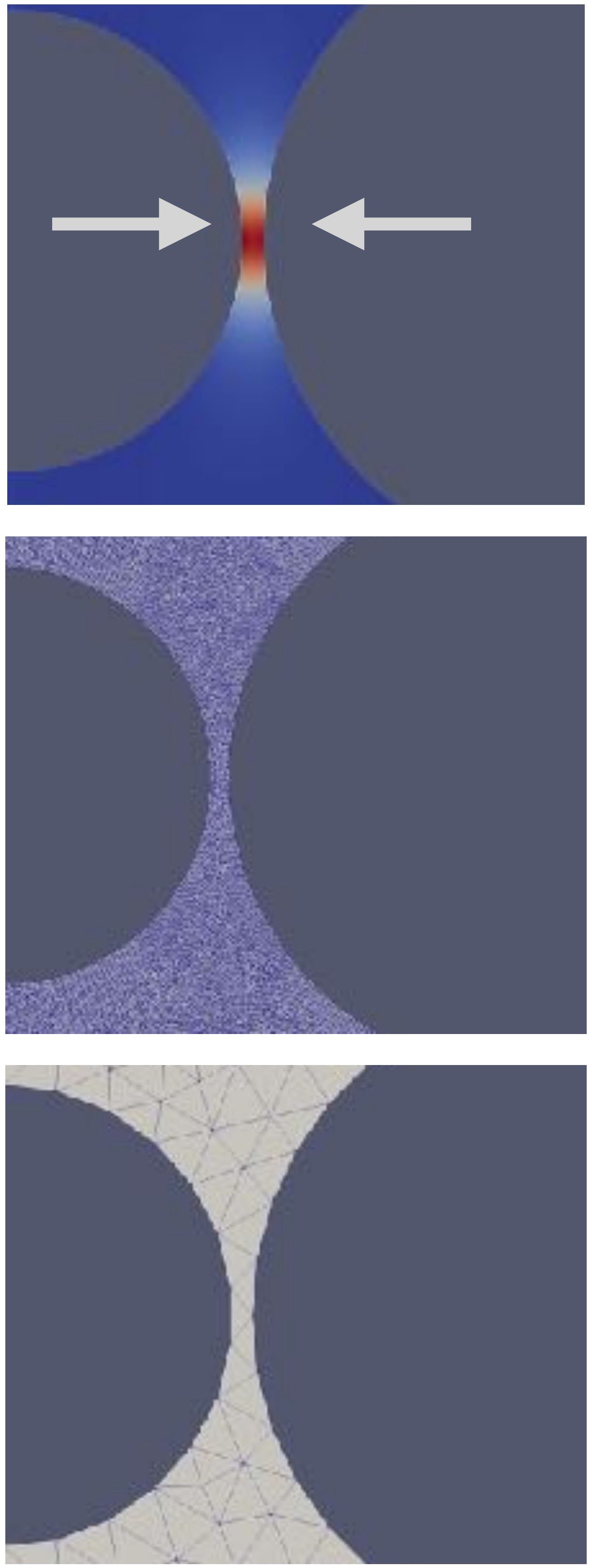
Lubrication force



$$F_{lub}(d) = -6\pi\mu r^2 \frac{\dot{d}}{d}$$

[Cox, 1974]

An explicit asymptotic expansion



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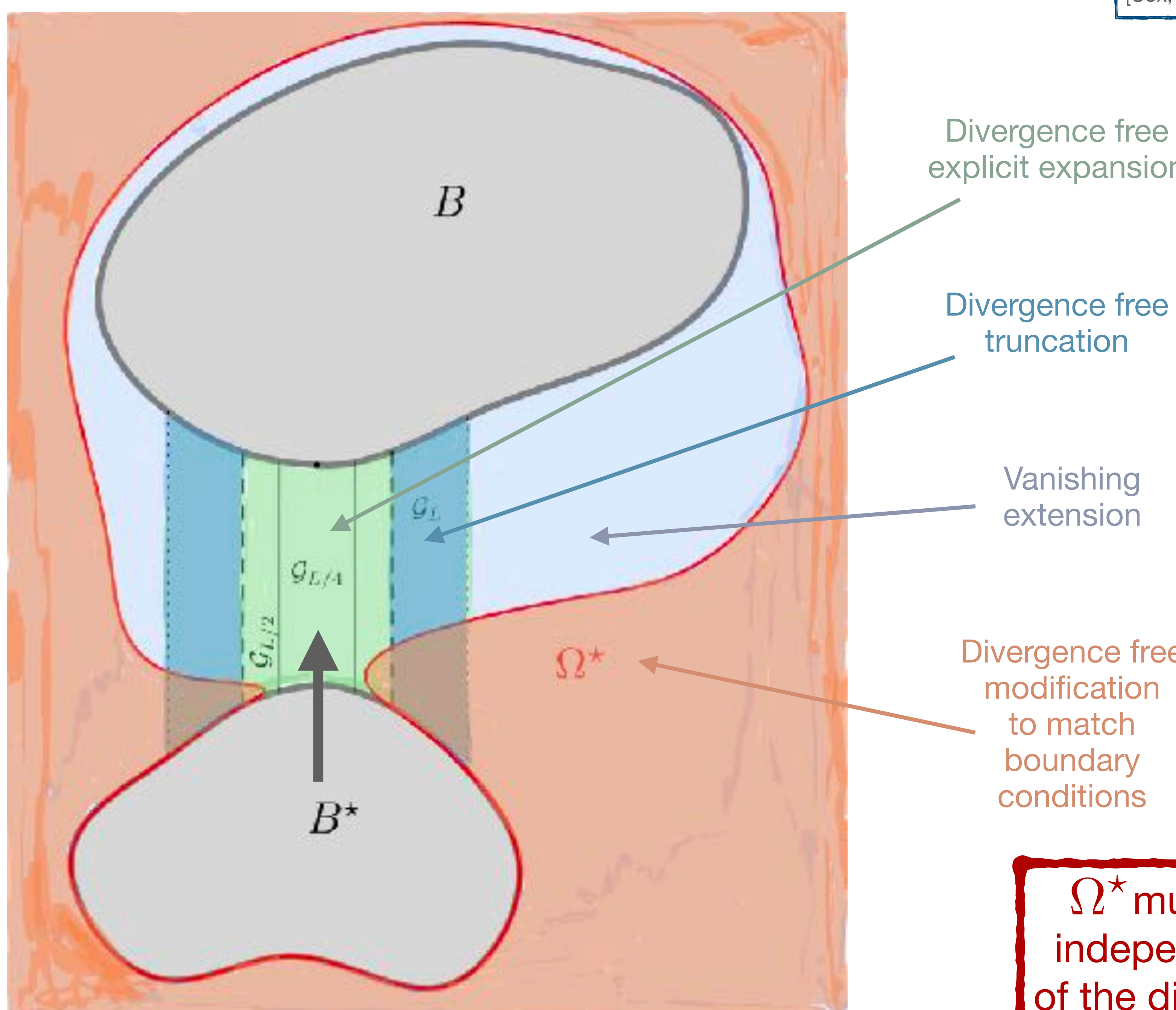
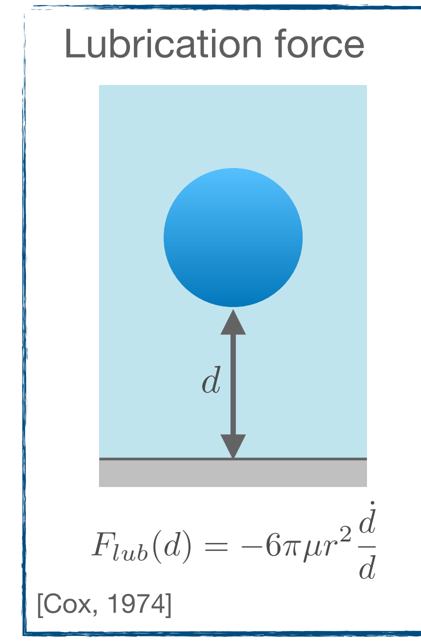
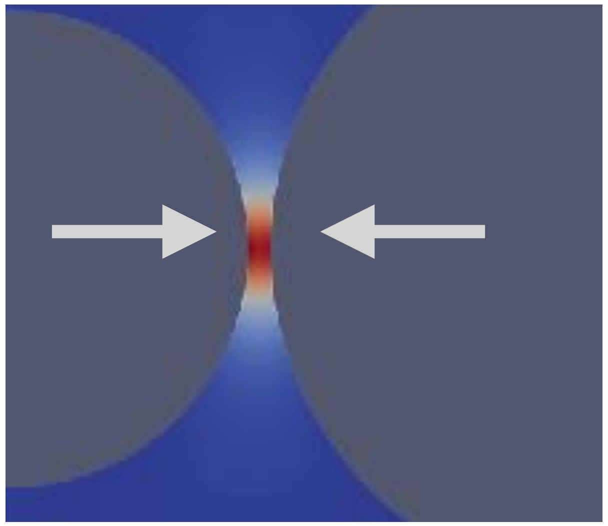
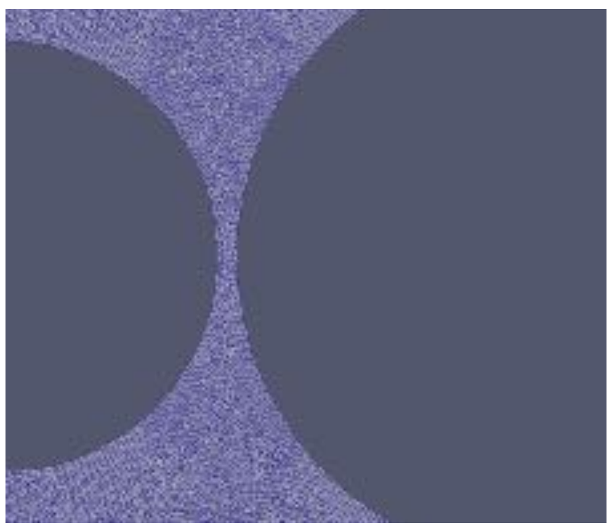


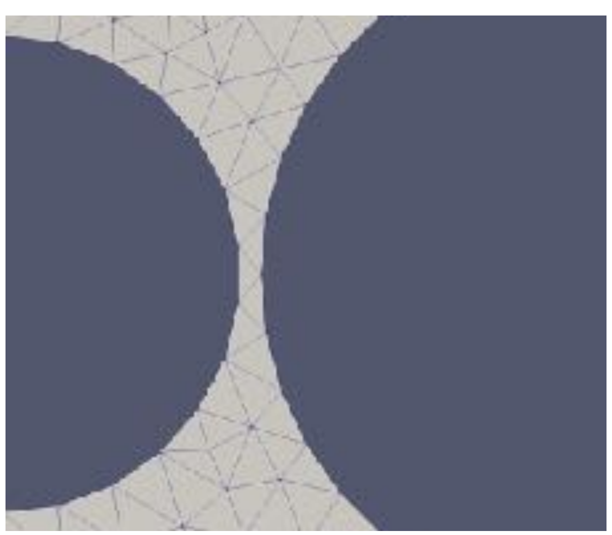


$\Omega^\star$ must be independent of the distance

An explicit asymptotic expansion







Explicit asymptotic expansion [Hillairet, Kelai, 2015]

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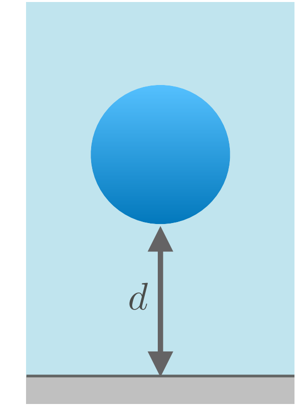
$(\mathbf{u}^{\mathrm{reg}}, \mathrm{p}^{\mathrm{reg}})$ solution to

$$\begin{aligned} -\mu\triangle\mathbf{u}^{\mathrm{reg}} + \nabla\mathrm{p}^{\mathrm{reg}} &= \mu\triangle\mathbf{u}^{\mathrm{sing}} - \nabla\mathrm{p}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \nabla\cdot\mathbf{u}^{\mathrm{reg}} &= -\nabla\cdot\mathbf{u}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \mathbf{u}^{\mathrm{reg}} &= \mathbf{u}^{\star} - \mathbf{u}^{\mathrm{sing}} && \text{on } \partial B \end{aligned}$$

Lubrication force



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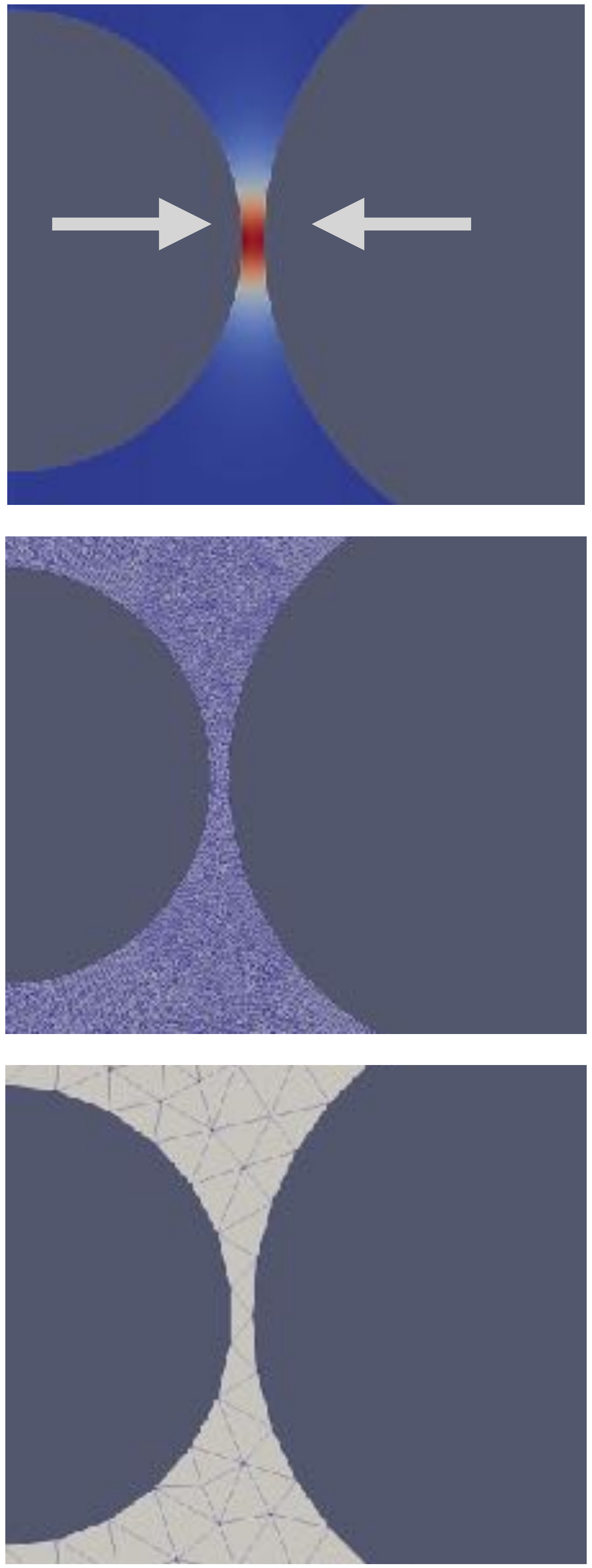
[Cox, 1974]

An explicit asymptotic expansion



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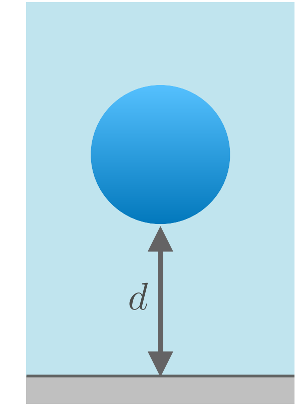
Lubrication force



$$F_{lub}(d) = -6\pi\mu r^2 \frac{\dot{d}}{d}$$

[Cox, 1974]

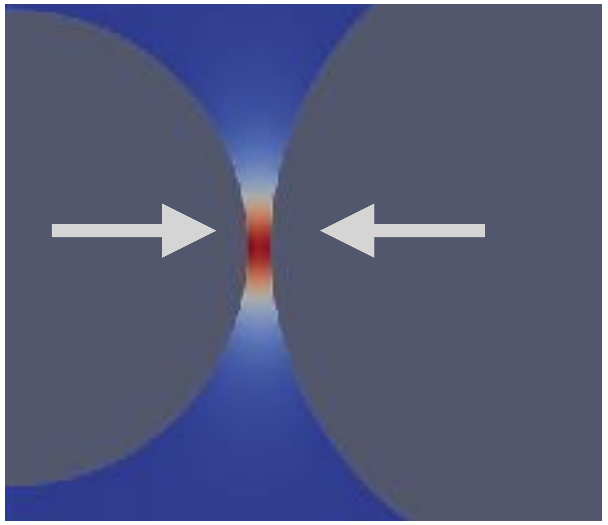
- [M. Hillairet, T. Kelai, 2015]

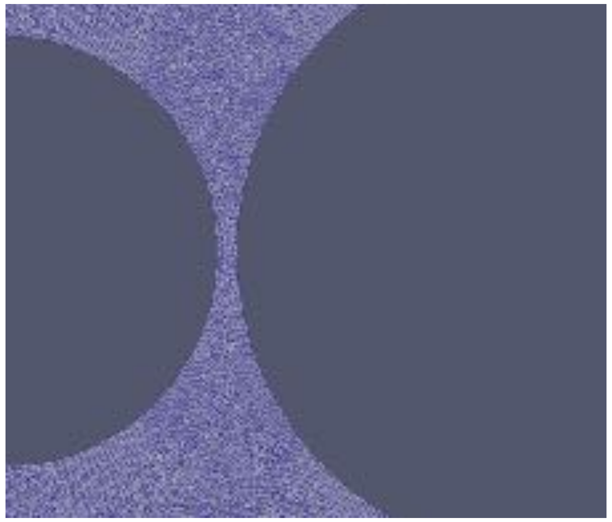
$$\|(\mathbf{u}^{\mathrm{reg}}, \mathrm{p}^{\mathrm{reg}})\| \leq C$$

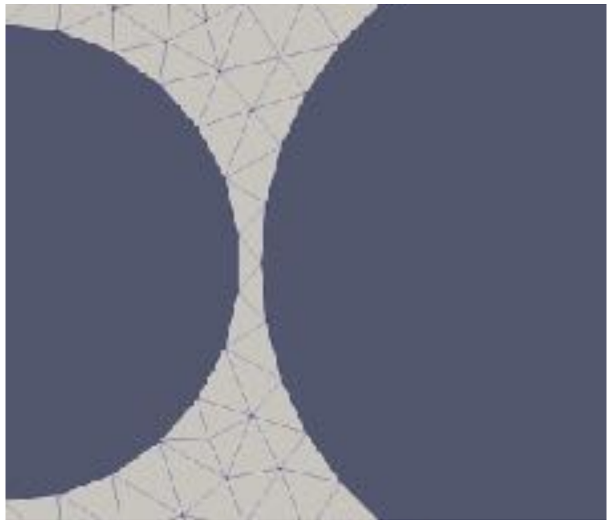
with C independent of the distance

- Hillairet, M., & Kelai, T. (2015). Justification of lubrication approximation: an application to fluid/solid interactions. Asymptotic Analysis, 95(3-4), 187-241.

Taking lubrication into account in the fluid solver.





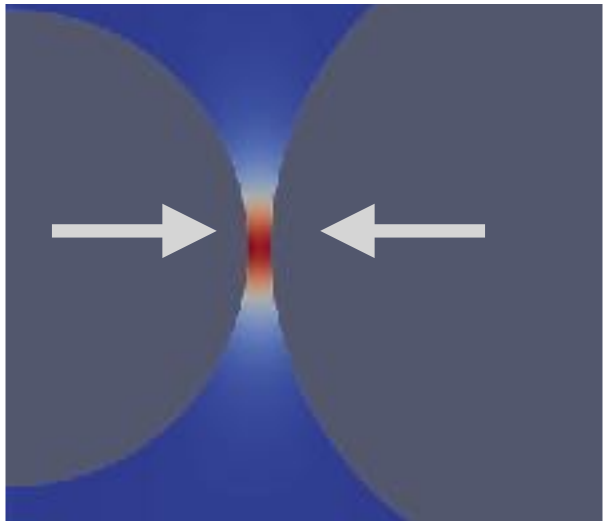


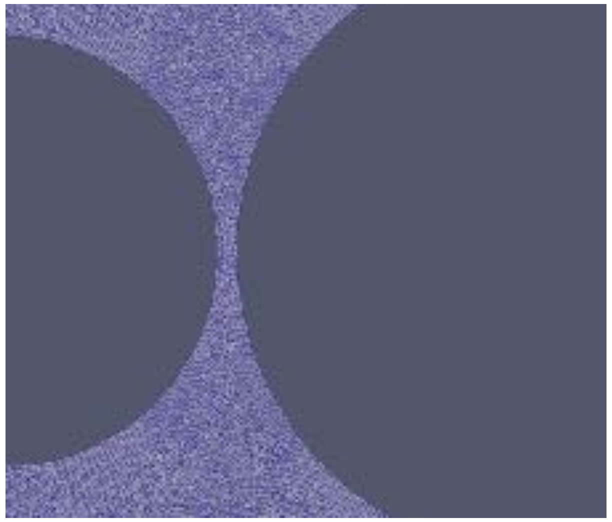
Adapted explicit asymptotic expansion

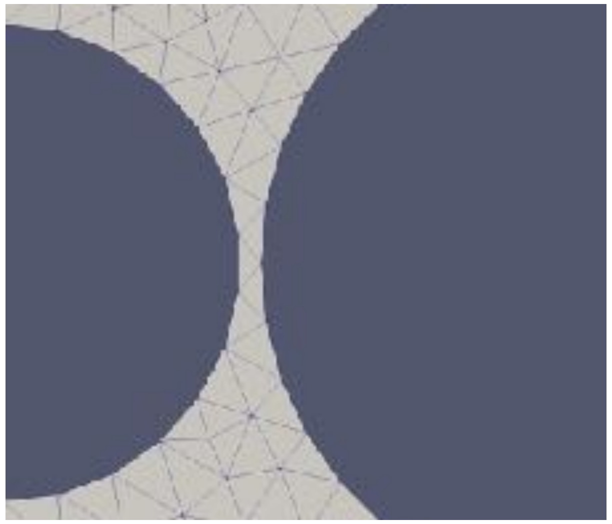
$$(\mathbf{u}_h^{\text{new}}, \mathrm{p}_h^{\text{new}}) = (\tilde{\mathbf{u}}^{\text{sing}}, \tilde{\mathrm{p}}^{\text{sing}}) + (\tilde{\mathbf{u}}_h^{\text{reg}}, \tilde{\mathrm{p}}_h^{\text{reg}})$$

• A. Lefebvre-Lepot et F. Nabet, Numerical simulation of rigid particles in Stokes flow : lubrication correction for general shape of particles, Math. Mode Nat. Phenom., 16 (2021), p. 45

Taking lubrication into account in the fluid solver.

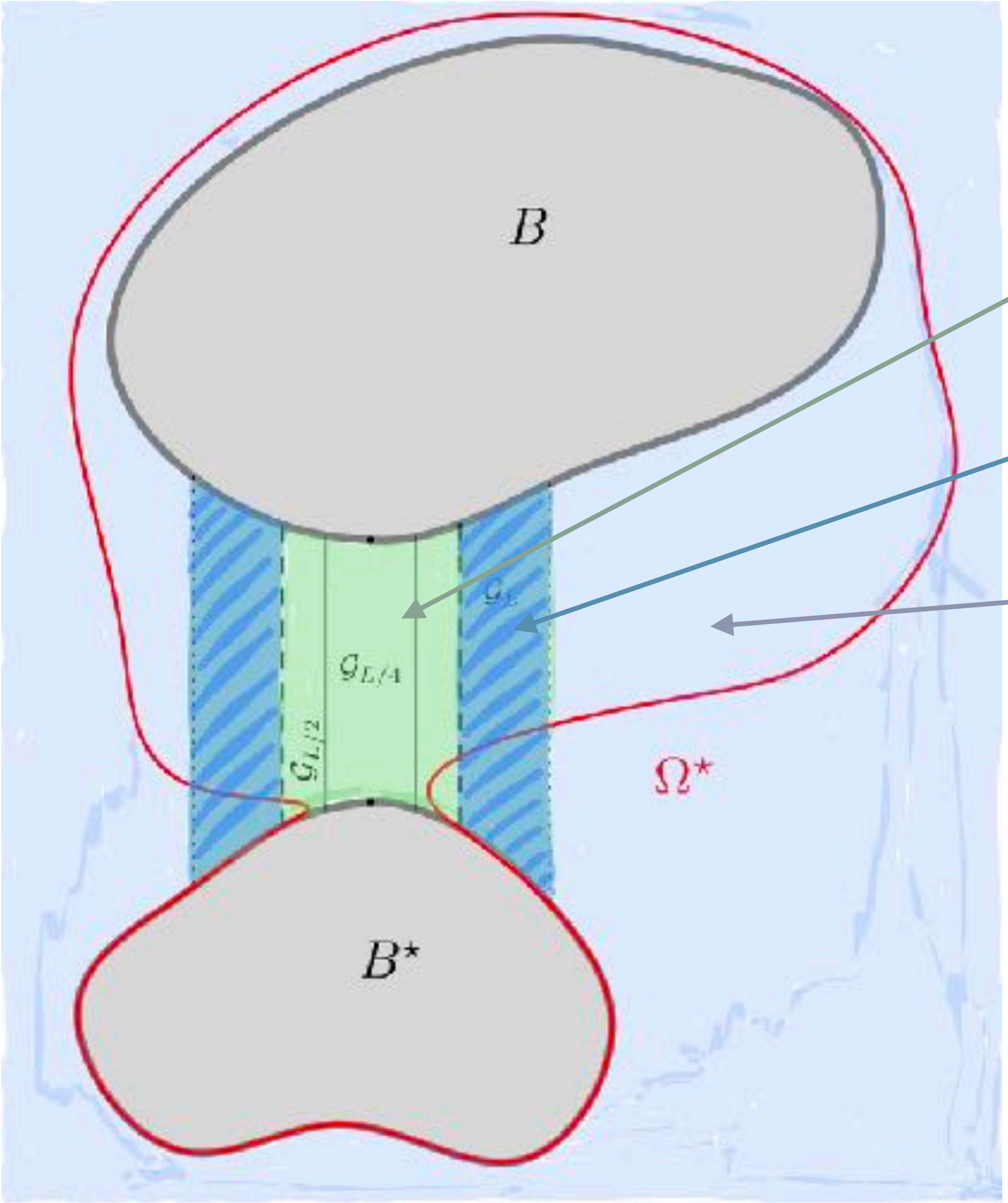






Adapted explicit asymptotic expansion

$$(\mathbf{u}_h^{\text{new}}, \mathrm{p}_h^{\text{new}}) = (\tilde{\mathbf{u}}^{\text{sing}}, \tilde{\mathrm{p}}^{\text{sing}}) + (\tilde{\mathbf{u}}_h^{\text{reg}}, \tilde{\mathrm{p}}_h^{\text{reg}})$$



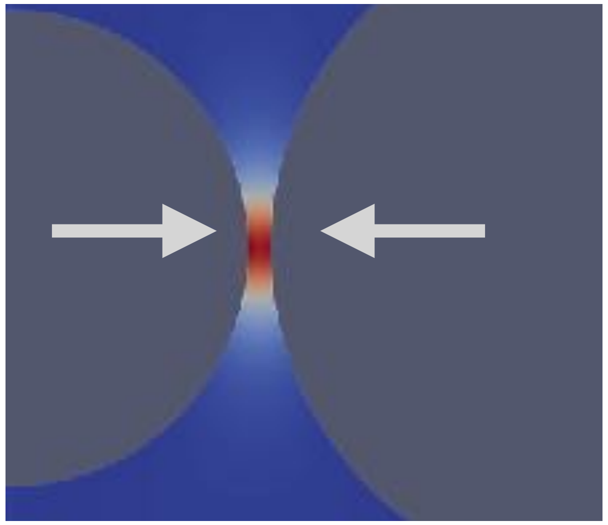
Divergence free explicit expansion

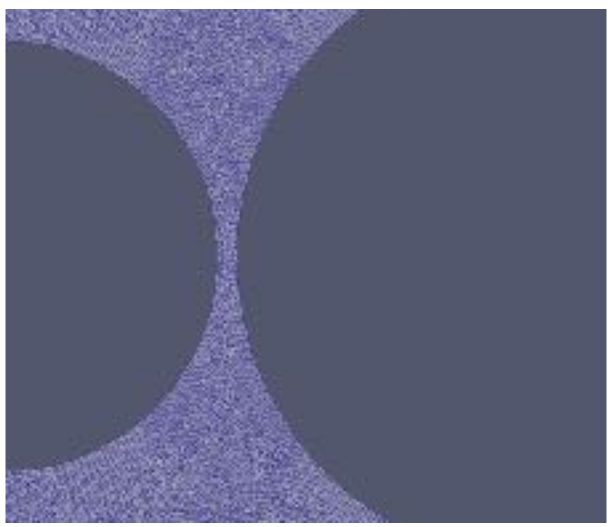
Modified non-divergence free truncation

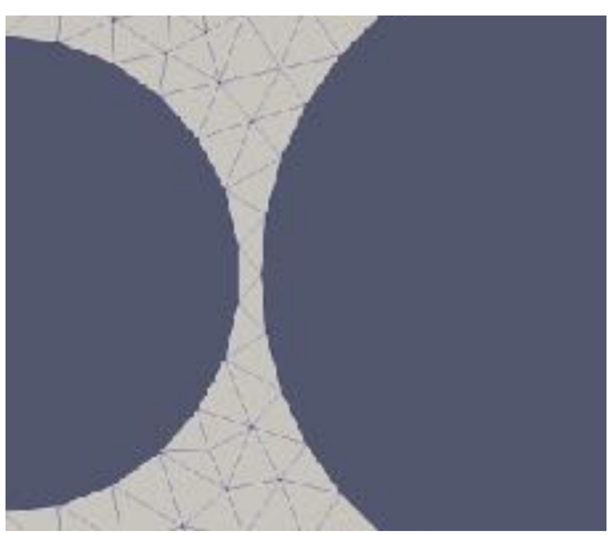
Vanishing extension

• A. Lefebvre-Lepot et F. Nabet, Numerical simulation of rigid particles in Stokes flow : lubrication correction for general shape of particles, Math. Mode
Nat. Phenom., 16 (2021), p. 45

Taking lubrication into account in the fluid solver.







Adapted explicit asymptotic expansion

$$(\mathbf{u}_h^{\mathrm{new}}, \mathrm{p}_h^{\mathrm{new}}) = (\tilde{\mathbf{u}}^{\mathrm{sing}}, \tilde{\mathrm{p}}^{\mathrm{sing}}) + (\tilde{\mathbf{u}}_h^{\mathrm{reg}}, \tilde{\mathrm{p}}_h^{\mathrm{reg}})$$

$(\tilde{\mathbf{u}}_h^{\mathrm{reg}}, \tilde{\mathrm{p}}_h^{\mathrm{reg}})$ numerical solution to

$$\begin{aligned} -\mu\triangle\tilde{\mathbf{u}}_h^{\mathrm{reg}} + \nabla\tilde{\mathrm{p}}_h^{\mathrm{reg}} &= \mu\triangle\tilde{\mathbf{u}}^{\mathrm{sing}} - \nabla\tilde{\mathrm{p}}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \nabla\cdot\tilde{\mathbf{u}}_h^{\mathrm{reg}} &= -\nabla\cdot\tilde{\mathbf{u}}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \tilde{\mathbf{u}}_h^{\mathrm{reg}} &= \mathbf{u}^\star - \tilde{\mathbf{u}}^{\mathrm{sing}} && \text{on } \partial B \end{aligned}$$

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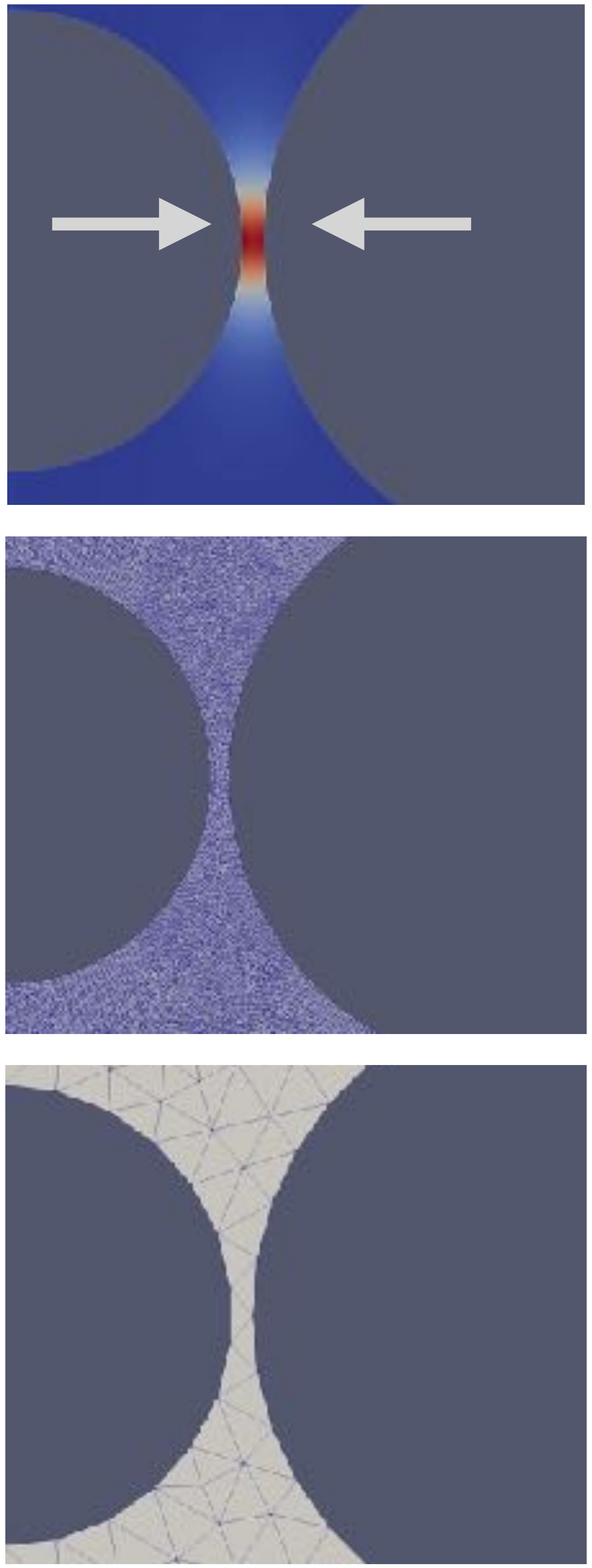
Taking lubrication into account in the fluid solver.



Adapted explicit asymptotic expansion

$$(\mathbf{u}_h^{\text{new}}, \mathrm{p}_h^{\text{new}}) = (\tilde{\mathbf{u}}^{\text{sing}}, \tilde{\mathrm{p}}^{\text{sing}}) + (\tilde{\mathbf{u}}_h^{\text{reg}}, \tilde{\mathrm{p}}_h^{\text{reg}})$$

$(\tilde{\mathbf{u}}_h^{\text{reg}}, \tilde{\mathrm{p}}_h^{\text{reg}})$ numerical solution to

$$\begin{aligned} -\mu\triangle\tilde{\mathbf{u}}_h^{\text{reg}} + \nabla\tilde{\mathrm{p}}_h^{\text{reg}} &= \mu\triangle\tilde{\mathbf{u}}^{\text{sing}} - \nabla\tilde{\mathrm{p}}^{\text{sing}} && \text{in } \mathcal{F} \\ \nabla\cdot\tilde{\mathbf{u}}_h^{\text{reg}} &= -\nabla\cdot\tilde{\mathbf{u}}^{\text{sing}} && \text{in } \mathcal{F} \\ \tilde{\mathbf{u}}_h^{\text{reg}} &= \mathbf{u}^\star - \tilde{\mathbf{u}}^{\text{sing}} && \text{on } \partial B \end{aligned}$$

$$\|\mathbf{u} - \mathbf{u}_h^{\text{new}}\| \leq C\, \|(\tilde{\mathbf{u}}^{\text{reg}}, \tilde{\mathrm{p}}^{\text{reg}})\|\, h^\alpha$$

Bounded independently of the distance

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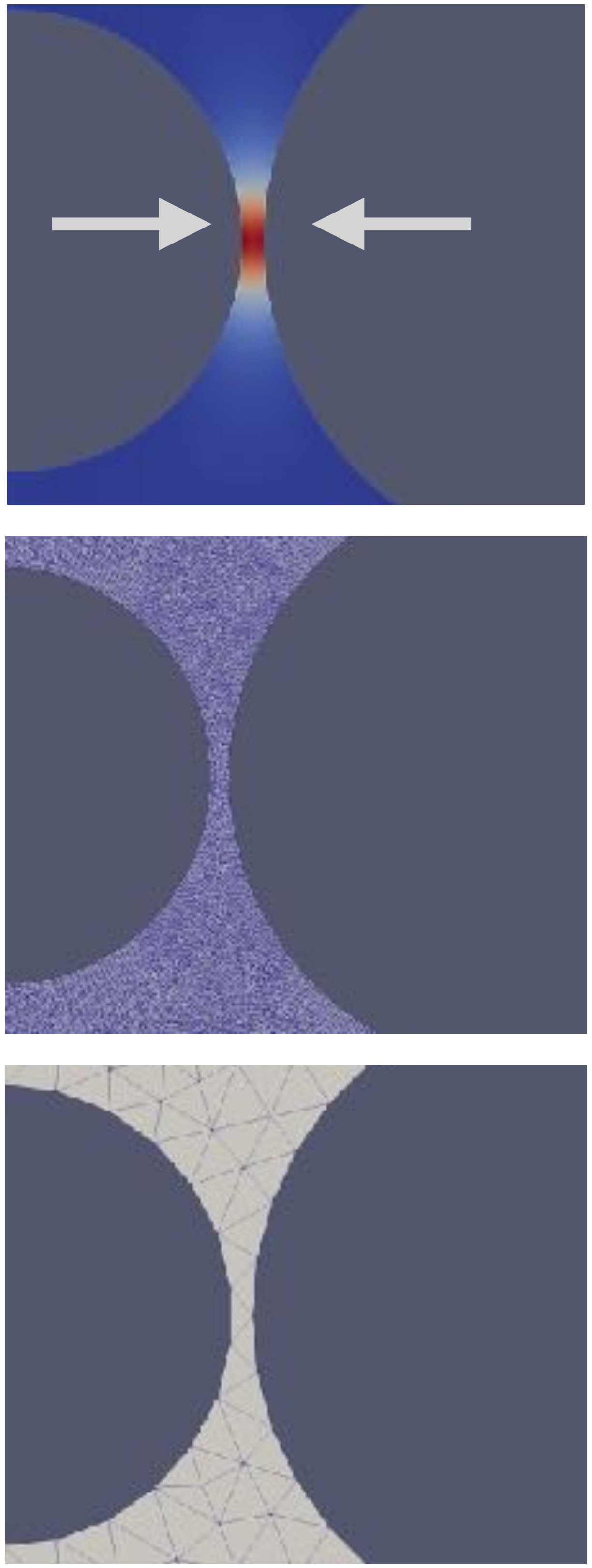


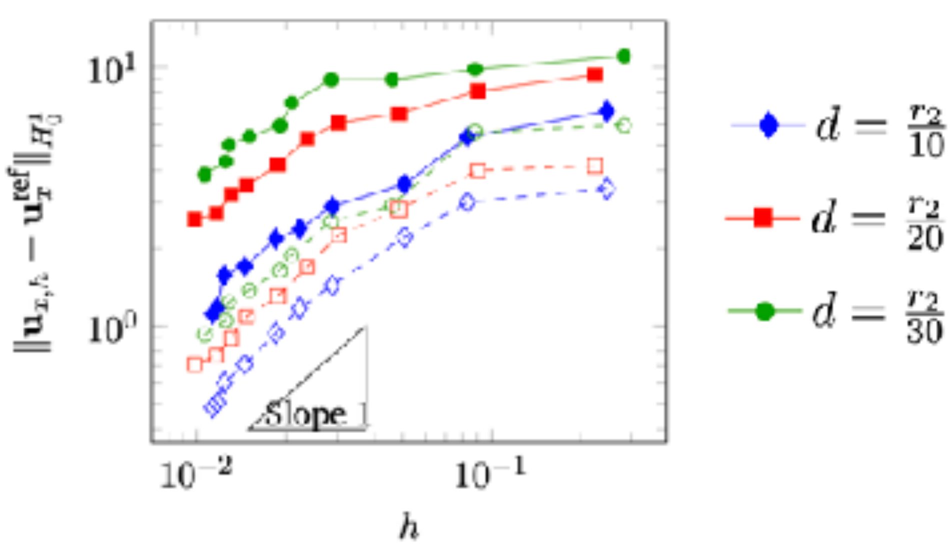
Adapted explicit asymptotic expansion

$$(\mathbf{u}_h^{\mathrm{new}}, \mathrm{p}_h^{\mathrm{new}}) = (\tilde{\mathbf{u}}^{\mathrm{sing}}, \tilde{\mathrm{p}}^{\mathrm{sing}}) + (\tilde{\mathbf{u}}_h^{\mathrm{reg}}, \tilde{\mathrm{p}}_h^{\mathrm{reg}})$$

$(\tilde{\mathbf{u}}_h^{\mathrm{reg}}, \tilde{\mathrm{p}}_h^{\mathrm{reg}})$ numerical solution to

$$\begin{aligned} -\mu\triangle\tilde{\mathbf{u}}_h^{\mathrm{reg}} + \nabla\tilde{\mathrm{p}}_h^{\mathrm{reg}} &= \mu\triangle\tilde{\mathbf{u}}^{\mathrm{sing}} - \nabla\tilde{\mathrm{p}}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \nabla\cdot\tilde{\mathbf{u}}_h^{\mathrm{reg}} &= -\nabla\cdot\tilde{\mathbf{u}}^{\mathrm{sing}} && \text{in } \mathcal{F} \\ \tilde{\mathbf{u}}_h^{\mathrm{reg}} &= \mathbf{u}^{\star} - \tilde{\mathbf{u}}^{\mathrm{sing}} && \text{on } \partial B \end{aligned}$$

$$\|\mathbf{u} - \mathbf{u}_h^{\mathrm{new}}\| \leq C\,\|(\tilde{\mathbf{u}}^{\mathrm{reg}}, \tilde{\mathrm{p}}^{\mathrm{reg}})\|\,h^{\alpha}$$



Bounded independently of the distance

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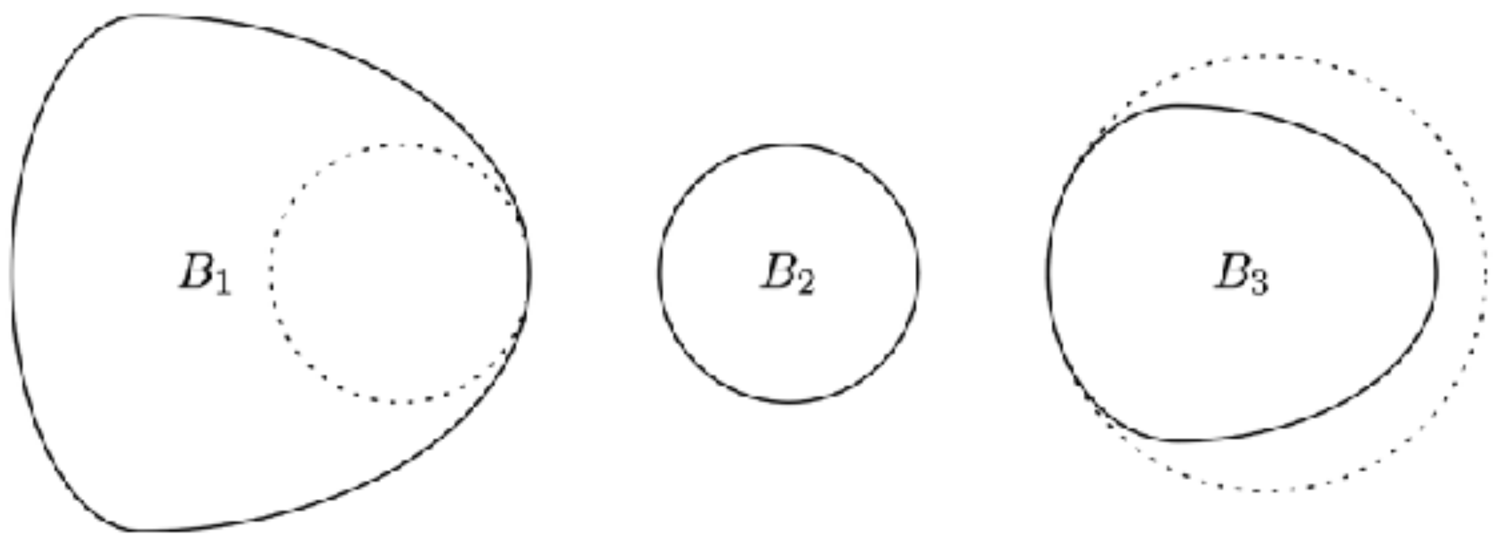
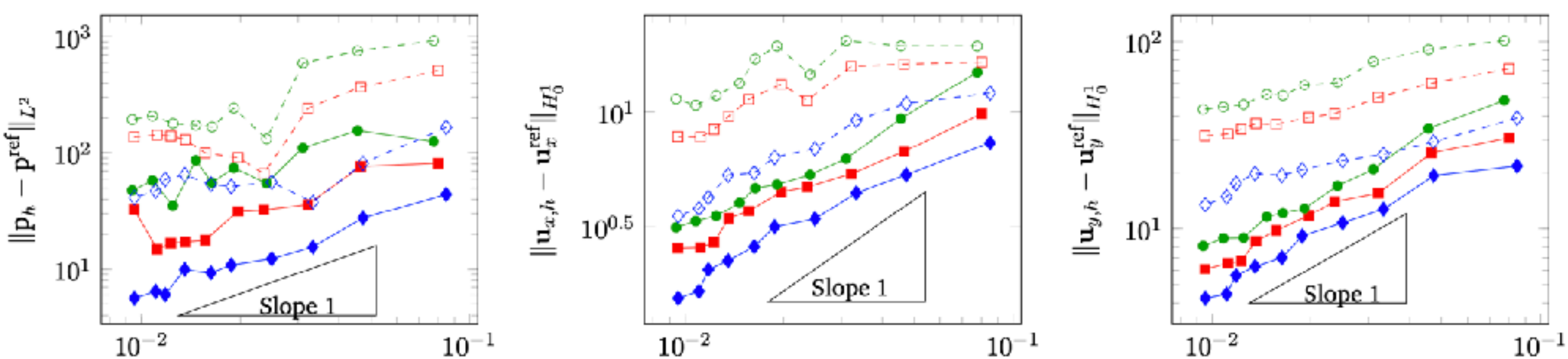
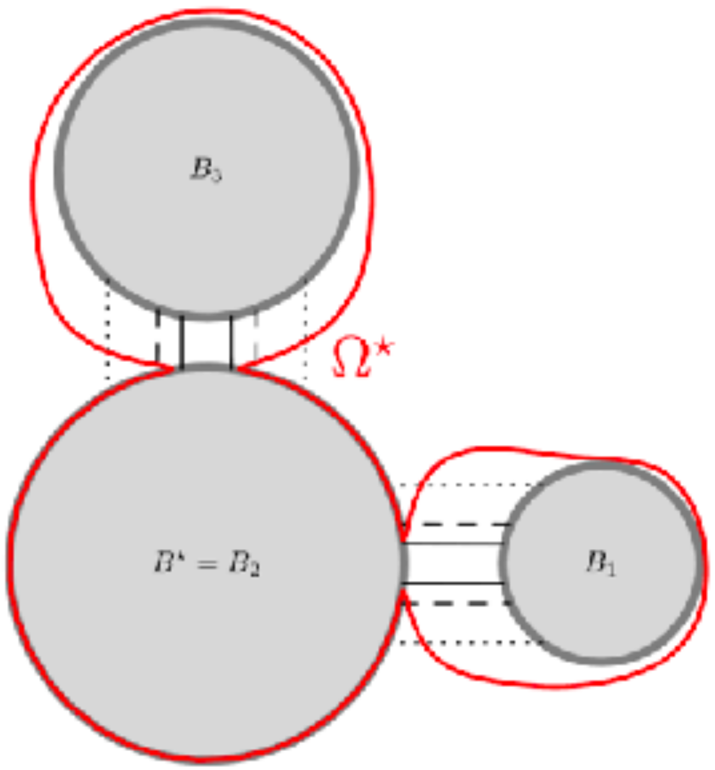
3 non-spherical particles with general rigid motion



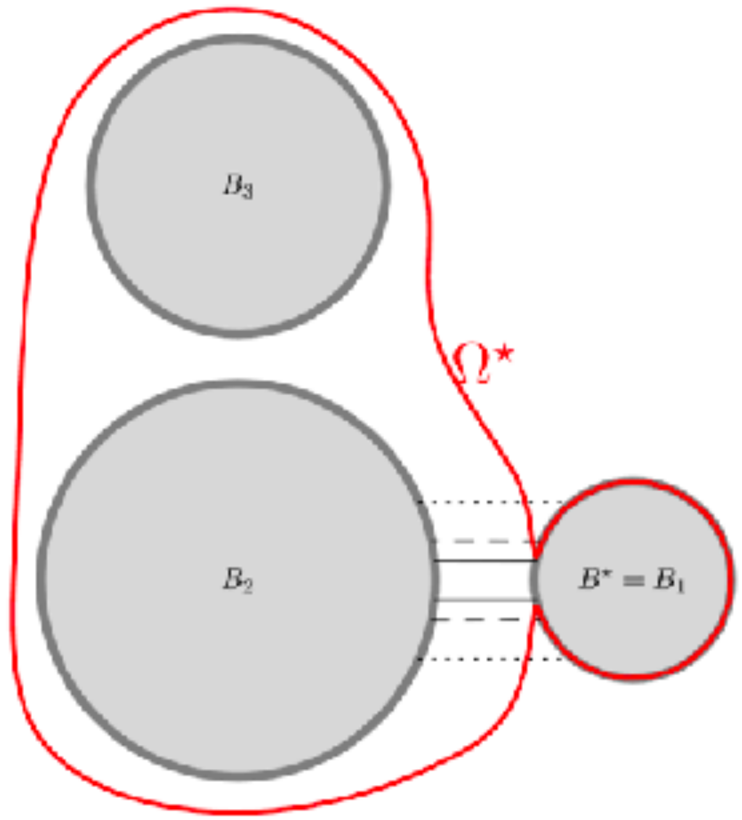
$$\mathbf{V}_1 = (1,5), \mathbf{V}_2 = (-1,-2), \mathbf{V}_3 = (1.5,-5), \mathbf{w}_1 = 3, \mathbf{w}_2 = 5, \mathbf{w}_3 = 10$$



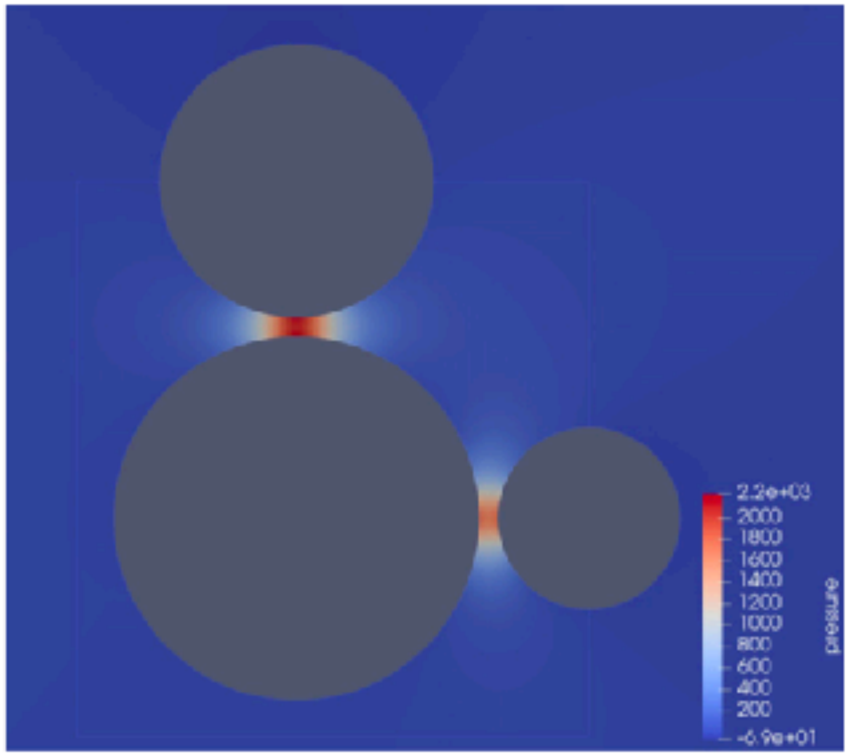
Limitation in dimension 2



(A) Choice for $\Omega^\star$ when B_2 is moving (2 singularities)

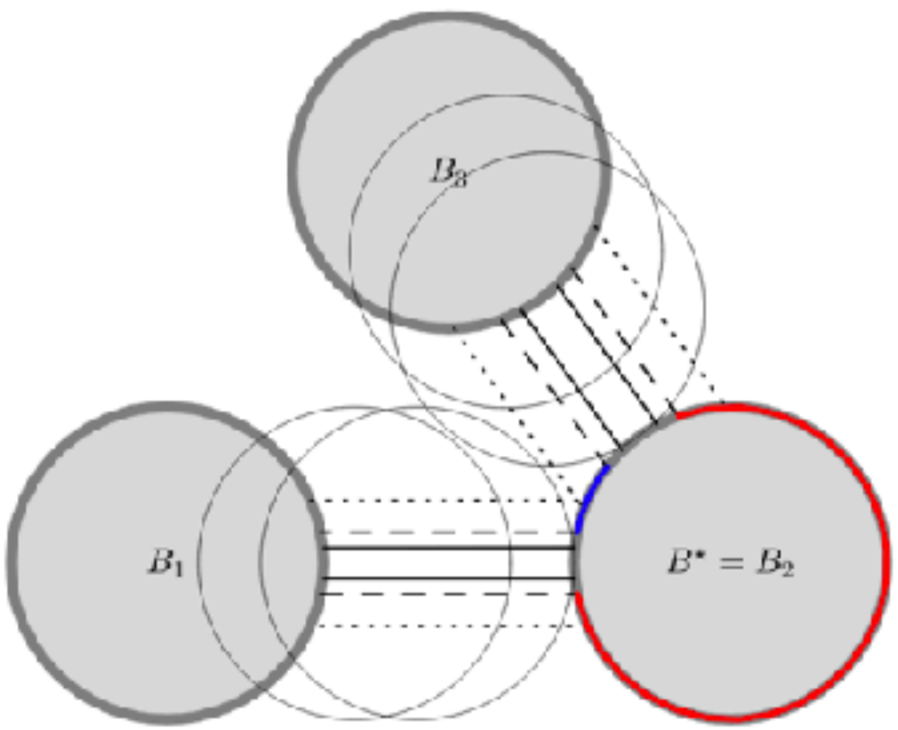


(B) Choice for $\Omega^\star$ when B_1 is moving (1 singularity)

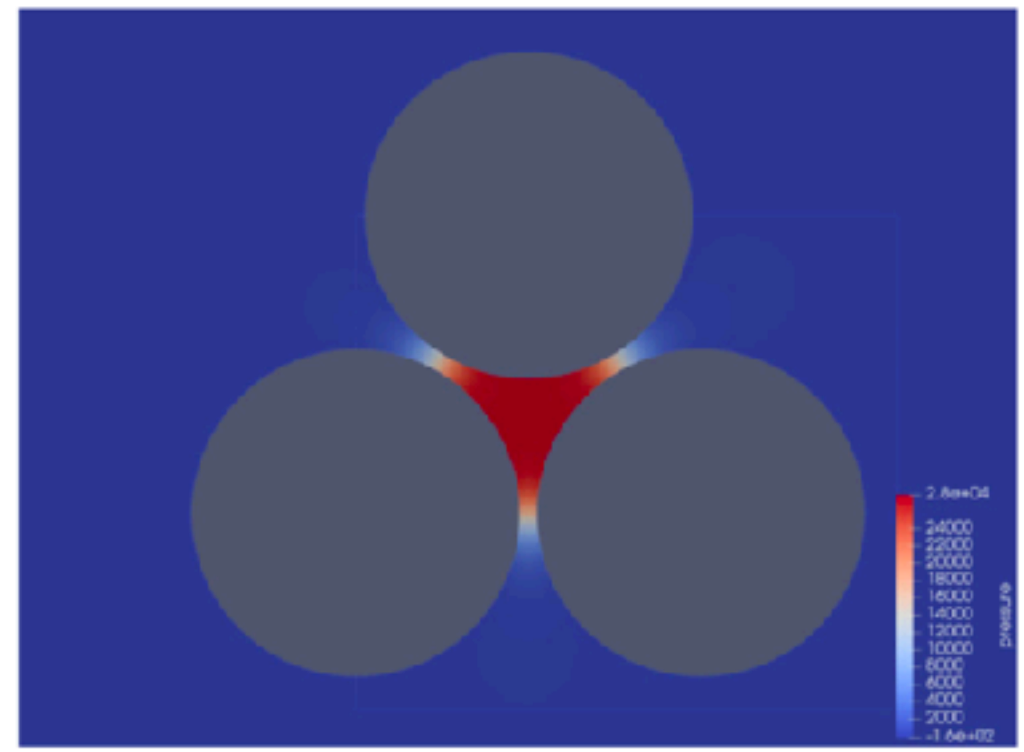


(C) Pressure field solution to the problem

Limitation in dimension 2



(A) Evolution of the configuration when the distance goes to zero



(B) Pressure field solution to the problem