Hyperbolic Systems:
Fluxes, Fluctuations and Computational Algorithms

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ABSTRACT

I first discuss the properties of solutions to hyperbolic balance laws and special physical situations that pose challenges in designing useful computational algorithms. The frameworks of finite volume (FV) and discontinuous Galerkin (DG) finite element methods are then formulated, from which the numerical flux and the numerical source emerge as key components to be specified. Fluxes fall into three distinct families: centred, upwind incomplete, and upwind complete Their main difference is revealed when resolving discontinuous intermediate characteristic fields. Fluxes for monotone (scalar case) first-order accurate schemes are briefly reviewed; these may also be utilized in the construction of high-order accurate schemes, both in the semidiscrete and fully discrete frameworks. The necessary non-linearity of high-order schemes, arising from Godunov's theorem, is also reviewed.
Fluxes for ADER high-order methods [1-6] are introduced, which are determined from the generalized Riemann problem (GRPm). This Cauchy problem has polynomials of degree *m* as initial conditions, and the equations may include stiff source terms. The resulting one-step, fully discrete schemes are of arbitrary (m+1)-th order accuracy in both space and time. Their spatial and temporal discretizations are intimately intertwined through the space-time dependence of the solution to the GRPm, for which there are currently seven methods to compute it [2,3,4]. For non-conservative systems, the analogues of numerical fluxes are fluctuations [6]. ADER operates in both the FV and DG frameworks [5], with reported implementations including schemes of up to 24th order of accuracy in both space and time. Through selected examples, we show that these methods are orders of magnitude cheaper than low-order methods for attaining a prescribed small error, making them essential for ambitious scientific applications. This lecture concludes with some applications.

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